TTP2
Lecture 17

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17 Basics of the Standard Model

We will discuss the current theory of subatomic world – the Standard Model of particle physics. This theory was formulated in a series of papers by Sh. Glashow, S. Weinberg and A. Salam. By the time the theory was proposed (late 1960s), it was known that there exists an electron (discovered in 1897) and an electron neutrino (1956), a muon (1936) and a muon neutrino (1962). The τ-lepton and the τ-neutrino were unknown.

It was also understood that there is an electromagnetic interaction, facilitated by massless photons, that can be described by a gauge theory known as Quantum Electrodynamics. It was also known that there are weak interactions that cause neutron decay $n \to p + e + \bar{\nu}_e$ and the muon decay $\mu \to e + \bar{\nu}_e + \nu_\mu$. There was Fermi theory, which stipulated that weak decays are described by the following Lagrangian

$$L_F = -\frac{G_F}{\sqrt{2}} \left[\bar{p}\gamma^\mu(1+\gamma_5)n\right] \left[\bar{e}\gamma^\mu(1+\gamma_5)\nu\right] + \text{h.c.}, \tag{17.1}$$

and a similar one for the muon decay. The Lagrangian Eq. (17.1) displays maximal parity violation in that only *left-handed fermions* ($\psi_L \sim (1+\gamma_5)\psi$) participate in weak interactions. Also, weak interactions were known to be *short-range*, at variance with electromagnetic interactions.

These two points imply that if weak interactions are to be described by gauge fields, these gauge fields have to couple differently to left- and right-handed fermions and, moreover, these gauge fields have to be massive to make sure that weak interactions are short-range.

The first point – a different role played by left and right fields in weak interactions – has important consequences. Indeed, a massive electron requires a term $L_m = m\bar{\psi}_L\psi_R + m\bar{\psi}_R\psi_L$ in the Lagrangian. If ψ_L and ψ_R transform differently under gauge transformations, the mass term in the Lagrangian will not be gauge-invariant. A possible way out is to re-use the idea of spontaneous symmetry breaking and apply it to fermions. Indeed, we start by considering a theory with *massless* fermions that couple to a scalar field, e.g. $L_m \to L_Y \sim \bar{\psi}_L\psi_R\varphi + \bar{\psi}_R\psi_L\varphi^\dagger$. The difference with the mass term is that now the field φ can also transform under gauge transformations and it may be possible to adjust quantum numbers of $\psi_{L,R}$ and φ in such a way that L_Y is invariant under gauge transformations. If, however, the field φ undergoes spontaneous symmetry breaking $\varphi \to v$, the Yukawa Lagrangian L_Y produces a mass term for the fermion ψ, i.e. $L_Y \to \bar{\psi}_L \psi_R m + \bar{\psi}_R \psi_L m$. So, similar to how gauge bosons get their masses in the process of spontaneous symmetry breaking, we can set up a gauge invariant theory with *massless* fermions that, after spontaneous symmetry breaking, turns into a theory with massive fermions. This is important since electrons and muons are, in fact, massive.

Let us now discuss how to couple fermions to gauge fields. If we put an electron and neutrino into an $SU(2)$ doublet and couple them to gauge fields, there will be terms in the Lagrangian that facilitate electron-to-neutrino transitions. If we look at the Fermi Lagrangian Eq. (17.1), we see that such transitions are present there. The same can be done for a muon and a muon neutrino. Since the Fermi Lagrangian only contains left-handed fields, we combine the left-handed electron and the left-handed neutrino into an $SU(2)$ (gauge) doublet

$$\Psi_L = \left(\begin{array}{c} \nu \\ e \end{array} \right)_L . \tag{17.2}$$

Since electrons are massive, we require a right-handed field e_R as well. This field does not participate in weak interactions since it is not part of the Fermi theory in Eq. (17.1). However, since QED is parity-conserving, photons do couple to left- and right-handed fields with equal strength. For this reason, we assume that *both* left-handed and right-handed fermions couple to an $U(1)$ field, but we cannot associate this field with the electromagnetic field right away (e.g. neutrinos do not couple to photons). We will also assume that electron neutrinos are massless and for this reason the right-handed neutrino field ν_R is not needed.

Therefore, we consider a theory based on the gauge group $SU_L(2) \times U_Y(1)$ and only consider electron and electron neutrino. Muon and muon neutrino can be added to the theory in a fully analogous way. Both Ψ_L and e_R transform under $U(1)$ but only Ψ_L transforms under $SU(2)$. We will have to break the symmetry to give masses to (some) gauge bosons and to the electron. We will do this with the help of a scalar complex doublet φ that transforms under both $SU(2)$ and $U(1)$. We will continue with writing down the Lagrangian for such a theory. We will call it $L_{\rm SM}$.

The first term in $L_{\rm SM}$, that is *completely fixed* once the gauge group is specified, is the kinetic term for gauge fields. We write

$$L_{\rm gauge} = L_{SU(2)} + L_{U(1)}, \tag{17.3}$$

where

$$L_{SU(2)} = -\frac{1}{4} W^i_{\mu\nu} W^{\mu\nu,i}, \qquad L_{U(1)} = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \tag{17.4}$$

with

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ikj} W^k_\mu W^j_\nu. \tag{17.5}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{17.6}$$

To move further, we need two covariant derivatives, one for the $SU(2)$ group and the other one for the $U(1)$ group. We have

$$D^{SU(2)}_\mu = \partial_\mu - igT^i W^i_\mu, \tag{17.7}$$

where T^i are generators of the $SU(2)$ Lie algebra. They satisfy

$$[T^a, T^b] = i\epsilon^{abc} T^c. \tag{17.8}$$

In case of an $SU(2)$ doublet, these generators are represented by $T^i = \tau^i/2$, where τ^i are the Pauli matrices. The $U(1)$ covariant derivative reads

$$D^{U(1)}_\mu = \partial_\mu - ig' Y B_\mu, \tag{17.9}$$

where Y defines a $U(1)$ charge (a "hypercharge") of a particular field in units of the fundamental $U(1)$ gauge coupling g'.

The Lagrangian L_{gauge} describes $3+1$ massless gauge bosons. The theory has to describe weak and electromagnetic interactions that require three massive (weak interactions, charged and neutral currents) bosons and one massless gauge boson (electromagnetism). As we know, this can be achieved by breaking *gauge* symmetry spontaneously. To this end, we introduce the Higgs field that is an $SU(2)$ doublet and has a $U(1)$ hypercharge Y_h. We write

$$L^{\text{kin}}_{\text{Higgs}} = (D_\mu \varphi)^\dagger (D^\mu \varphi), \tag{17.10}$$

where

$$\varphi = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}, \tag{17.11}$$

and where

$$D_\mu = \partial_\mu - igT^i W^i_\mu - ig' B_\mu Y_h. \tag{17.12}$$

The part of the Higgs Lagrangian that is responsible for breaking the symmetry is

$$L_{\rm EWSB} = -\frac{\lambda}{4}\left(\varphi^\dagger\varphi - \frac{v^2}{2}\right)^2. \tag{17.13}$$

The EWSB Lagrangian requires that we choose the non-vanishing vacuum field. We write

$$\varphi(x) = \left(\begin{array}{c} 0 \\ \frac{v+h(x)}{\sqrt{2}} \end{array}\right) \tag{17.14}$$

We know that this is a complete parameterization of the doublet after the symmetry breaking since the rest can be removed by a gauge transformation.

Let us compute the mass spectrum of gauge bosons in such a theory. The mass spectrum follows from $L_{\rm Higgs}^{\rm kin}$ upon substituting $\varphi \to \varphi_{\rm vac}$ there. We find

$$L_{\rm vac}^{\rm kin} \to \varphi_{\rm vac}^T\left[igW_\mu^i T^i + ig'B_\mu Y_h\right]\left[-igW^{\mu,j}T^j - ig'B^\mu Y_h\right]\varphi_{\rm vac}, \tag{17.15}$$

where

$$\varphi_{\rm vac} = \left(\begin{array}{c} 0 \\ \frac{v}{\sqrt{2}} \end{array}\right) \tag{17.16}$$

We expand Eq. (17.15) and write

$$\varphi_{\rm vac}^T\left[g^2 W_\mu^i W^{j,\mu}T^iT^j + Y_h^2 g'^2 B_\mu B^\mu + 2gg'Y_h W_\mu^i B^\mu T^i\right]\varphi_{\rm vac}. \tag{17.17}$$

Then, we compute

$$\begin{aligned}
&g^2 W_\mu^i W^{j,\mu}\ \varphi_{\rm vac}^T\tau^i\tau^j\varphi_{\rm vac} = \frac{g^2}{4}W_\mu^i W^{j,\mu}\varphi_{\rm vac}^T\delta_{ij}\hat{1}\varphi_{\rm vac} = \frac{g^2v^2}{8}W_\mu^i W^{i,\mu}, \\
&Y_h^2 g'^2 B_\mu B^\mu \varphi_{\rm vac}^T\varphi_{\rm vac} = \frac{Y_h^2 g'^2 v^2}{2}B_\mu B^\mu, \\
&2gg'Y_h W_\mu^i B^\mu \varphi_{\rm vac}^T\tau^i\varphi_{\rm vac} = -\frac{gg'Y_h v^2}{2}W_\mu^i B^\mu \delta^{i3},
\end{aligned} \tag{17.18}$$

so that

$$\begin{aligned}
L_{\rm vac}^{\rm kin} &\to \frac{g^2v^2}{8}\,W_\mu^i W^{i,\mu} + \frac{Y_h^2 g'^2 v^2}{2}B_\mu B^\mu - \frac{gg'Y_h v^2}{2}W_\mu^3 B^\mu \\
&= \frac{v^2g^2}{8}\left(W_\mu^1 W^{1,\mu} + W_\mu^2 W^{2,\mu}\right) \\
&\quad + \frac{v^2(g^2+4Y_h^2g'^2)}{8}\left(\frac{g}{\sqrt{g^2+4Y_h^2g'^2}}W_\mu^3 - \frac{2g'Y_h}{\sqrt{g^2+4Y_h^2g'^2}}B_\mu\right)^2.
\end{aligned} \tag{17.19}$$

It follows from Eq. (17.19) that the two fields W^1, W^2 acquire the mass

$$m_{1,2} = \frac{gv}{2}, \tag{17.20}$$

whereas a combination of W^3 and B fields

$$Z_\mu = \cos\theta \, W_\mu^{(3)} - \sin\theta \; B_\mu \tag{17.21}$$

acquires the mass

$$m_Z = \frac{vg}{2\cos\theta}. \tag{17.22}$$

To write the above equations in a compact form, we introduced weak mixing angle θ; the cosine and sine of this angle is fixed in terms of the gauge coupling and the Higgs boson hypercharge

$$\cos\theta = \frac{g}{\sqrt{g^2 + 4Y_h^2 g'^2}}, \qquad \sin\theta = \frac{2g'Y_h}{\sqrt{g^2 + 4Y_h^2 g'^2}}. \tag{17.23}$$

A combination of fields that is orthogonal to Eq. (17.21) reads

$$A_\mu = \sin\theta \, W_\mu^{(3)} + \cos\theta \; B_\mu. \tag{17.24}$$

An important consequence of Eq. (17.19) is that the field A_μ remains massless; for this reason, we would like to associate it with the photon.

For future reference, we write formulas for the inverse field transformation

$$\begin{aligned} W_\mu^3 &= \cos\theta Z_\mu + \sin\theta A_\mu, \\ B_\mu &= -\sin\theta Z_\mu + \cos\theta A_\mu. \end{aligned} \tag{17.25}$$

It is convenient to choose $Y_h = 1/2$. In this case, formulas for the mixing angles become very simple

$$\cos\theta = \frac{g}{\sqrt{g^2 + g'^2}}, \qquad \sin\theta = \frac{g'}{\sqrt{g^2 + g'^2}}, \tag{17.26}$$

and the combination of fields that remains massless couples to the Higgs doublet with the following strength

$$D_\mu \phi \to \left(-igT^3 W_\mu^3 - ig'YB_\mu\right)\phi \to -ig\cos\theta A_\mu \left(T_3 + Y\right)\phi. \tag{17.27}$$

Since for the choice of the Higgs field as in Eq. (17.15)

$$T_3\varphi = -\frac{1}{2}\,\varphi, \tag{17.28}$$

and since we have chosen $Y\varphi = 1/2\varphi$,

$$(T_3 + Y)\varphi = 0, \tag{17.29}$$

and the Higgs field does not couple to photons. Since photons couple to the electric charge, we conclude that the Higgs field (including the vacuum expectation value) is neutral.

We continue with the discussion on leptons. We will only consider electron and electron neutrino since muon and muon neutrino are included into the SM in an identical way. As we already said, the left-handed fields are $SU(2)$ doublets and the right-handed electrons are $SU(2)$ singlets. We consider massless neutrinos and, therefore, we do not introduce the right-handed neutrino field. We write the Lagrangian

$$L_{\mathsf{F}} = \bar{\psi}_L i\hat{D}_L\psi_L + \bar{e}_R i\hat{D}_R e_R, \tag{17.30}$$

where

$$\begin{aligned} D_L^\mu &= \partial^\mu - igW_\mu^i\frac{\tau^i}{2} - ig'B_\mu Y_L, \\ D_R^\mu &= \partial^\mu - ig'B_\mu Y_R. \end{aligned} \tag{17.31}$$

Since left and right fields have different quantum numbers, we cannot write the mass term for an electron since it mixes left- and right-handed fields. We will return to this question after we study how gauge bosons interact with fermions.

To understand this, we neglect partial derivatives in Eq. (17.31) and consider only terms of the type $\bar{\psi}V_\mu\gamma^\mu\psi = \bar{\psi}\hat{V}\psi$, where V_μ is a gauge field. Then

$$\begin{aligned} \bar{\psi}_L i\hat{D}_L\psi_L &\to \frac{1}{2}\left[\bar{\nu}_L(g\hat{W}^3 + 2g'\hat{B}Y_L)\nu_L + \bar{e}_L(-g\hat{W}^3 + 2g'\hat{B}Y_L)e_L \right. \\ &\qquad \left. + g\bar{\nu}_L(\hat{W}^1 - i\hat{W}^2)e_L + g\bar{e}_L(\hat{W}^1 + i\hat{W}^2)\nu_L\right], \\ \bar{e}_R i\hat{D}_R e_R &\to g'Y_R\bar{e}_R\hat{B}e_R. \end{aligned} \tag{17.32}$$

To understand how electrons and neutrinos interact with gauge fields, we express W_μ^3 and B_μ through mass eigenstates of gauge fields.

We begin by using Eq. (17.25) to determine how Z_μ and A_μ interact with electrons and neutrinos. Of particular importance to us is the coupling of the photon field A_μ to fermions. This is so because we know how this coupling should look like. In particular, the photon should *not* couple to neutrinos since they have no electric charge and it should couple to both left- and right-handed electrons with equal strength that is proportional to the electron's electric charge. These features provide important constraints for the theory.

We begin with neutrino's couplings to photons. This coupling arises from the $\bar{\nu}_L \cdots \nu_L$ term in Eq. (17.32) if we replace both W_μ^3 and B_μ with A_μ, following Eq. (17.25). We find

$$\frac{1}{2}\bar{\nu}_L(g\hat{W}^3+2g'\hat{B}Y_L)\nu_L \to \frac{1}{2}(g\sin\theta+2g'Y_L\cos\theta)\bar{\nu}_L\hat{A}\nu_L = \frac{gg'\left(1+2Y_L\right)}{\sqrt{g^2+g'^2}}\bar{\nu}_L\hat{A}\nu_L. \tag{17.33}$$

Hence, to ensure that photons do not couple to neutrinos, we need to choose the hypercharge of left-handed leptons to be $Y_L = -1/2$. We can check that with this choice the relation between T_3 and the hypercharge operator

$$Q = T_3 + Y \tag{17.34}$$

predicts that the electric charge of the left-handed electron is -1 and the charge of neutrino is zero.

The hypercharge of the right-handed electron follows from the requirement that photons couple to both left- and right-handed electrons in the same way. We write

$$\frac{1}{2}\bar{e}_L(-g\hat{W}^3+2g'\hat{B}Y_L)e_L + g'Y_R\bar{e}_R\hat{B}e_R \to -g\sin\theta\bar{e}_L\hat{A}e_L + g'\cos\theta Y_R\bar{e}_R\hat{A}e_R \tag{17.35}$$

Since $g\sin\theta = g'\cos\theta$, it follows from the above equation that $Y_R = -1$. Also since the coupling of electrons to photons is proportional to the electric charge, we find

$$g\sin\theta = g'\cos\theta = e = \sqrt{4\pi\alpha}. \tag{17.36}$$

We can now easily write the remaining couplings of gauge bosons to leptons. From Eq. (17.32) we have distinct contributions:

$$
\begin{aligned}
&\text{Charged current interaction :} && \frac{g}{2\sqrt{2}}\bar{\nu}\gamma_\mu(1+\gamma_5)eW^{\mu,+} + h.c.;\\
&\text{Neutral current interaction :} && \frac{g}{4\cos\theta}\bar{\nu}\gamma_\mu(1+\gamma_5)\nu Z^\mu;\\
&\text{Neutral current interaction :} && -\frac{g}{4\cos\theta}\bar{e}\left[\gamma_\mu(1-4\sin^2\theta)+\gamma_\mu\gamma_5\right]eZ^\mu;\\
&\text{Electromagnetic interaction :} && -e\,\bar{e}\gamma_\mu eA^\mu.
\end{aligned}
$$

We will discuss in detail how to introduce fermion masses into the theory in the next lecture. Here we will very briefly say a few words about how this can be done. Since left- and right-handed fields have different quantum numbers, the relevant term also involves the Higgs field. We write

$$L_Y = \left[f\bar{\psi}_L\varphi e_R + h.c.\right], \tag{17.37}$$

The Yukawa Lagrangian in Eq. (17.37) is obviously invariant under $SU(2)$ transformations since *both*, the left-handed field and the Higgs field are $SU(2)$ doublets. We can also check that it is $U(1)$ invariant since the following condition is satisfied

$$Y_H + Y_R - Y_L = 0 \tag{17.38}$$

It is then easy to see that the Yukawa Lagrangian Eq. (17.37) provides a mass to electrons after the symmetry breaking. Indeed, upon replacing the Higgs field with its vacuum expectation value we obtain

$$L_Y \to f_e\left[\bar{\varphi}_L\varphi_{\rm vac}e_R + h.c.\right] = \frac{f_e v}{\sqrt{2}}\left[\bar{e}_L e_R + h.c.\right] = m_e\left[\bar{e}_L e_R + h.c.\right], \tag{17.39}$$

where $m_e = f_e v/\sqrt{2}$ is the electron mass. As we see, neutrino remains massless.