TTP2
Lecture 19

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19 Quantization of gauge theories with spontaneous symmetry breaking

We would like to discuss quantization of theories with spontaneously broken gauge symmetries since there are some peculiarities that one needs to be aware of. We will start with a a very simple $U(1)$ gauge theory coupled to a complex scalar field which develops a non-vanishing vacuum expectation value.

The Lagrangian of the theory reads

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^+(D^\mu\phi) - V(\phi), \tag{19.1}$$

where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative, $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and

$$V(\phi) = -\frac{\lambda}{4}\left(|\phi|^2 - \frac{v^2}{2}\right)^2. \tag{19.2}$$

We then choose the vacuum expectation value such that

$$\langle 0|\phi_1|0\rangle = v, \quad \langle 0|\phi_2|0\rangle = 0, \tag{19.3}$$

and write

$$\phi_1 = v + h, \tag{19.4}$$

to describe deviations from the vacuum expectation value. We find

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial_\mu h + gA_\mu\phi_2\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2 - gA_\mu(v+h)\right)^2 - V(\phi). \tag{19.5}$$

To identify propagators and interaction vertices we separate terms in the Lagrangian that are quadratic in the fields from the rest that we interpret as interactions. If we do this, we will find that in the third term in the Lagrangian there is a contribution $-gv\partial_\mu\phi_2A^\mu$ that "mixes" the field ϕ_2 and the field A_μ. In our previous discussion of such theories we were removing such terms by performing a gauge transformation on the field A_μ. However, this discussion was performed at the level of a classical field theory. If we attempt to properly quantize a gauge theory using the path integral method, we must fix the gauge by inserting a δ-functional into the integral over all field configurations. We should try to do the same also for in this case as well.

To quantize the original theory, we write the path integral as follows

$$Z = \int \mathcal{D}\phi \mathcal{D}A_\mu \, \mathcal{D}\chi \;\; \delta(G[A^\chi]) \det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right) e^{iS[A,\phi]}. \tag{19.6}$$

The gauge fixing condition that we choose reads

$$\partial^\mu A_\mu - g(x) = 0. \tag{19.7}$$

Anticipating the transition to a spontaneously broken theory , we choose

$$g(x) = -\xi g v \phi_2 + f(x), \tag{19.8}$$

where $f(x)$ is an arbitrary function and ξ is the parameter. Since the result does not depend on the choice of $g(x)$ or on the choice of $f(x)$, we write

$$\begin{aligned} Z &= \int \mathcal{D}f \mathcal{D}\phi \mathcal{D}A_\mu \det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right) \delta(\partial^\mu A_\mu + \xi g v \phi_2 - f(x)) e^{iS[A,\phi] - i\int \mathrm{d}^4x f(x)^2/2\xi} \\ &= \int \mathcal{D}\phi \mathcal{D}A_\mu \det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right) e^{iS_\xi[A,\phi]}, \end{aligned} \tag{19.9}$$

where the new action S_ξ reads

$$S_\xi = \int \mathrm{d}^4x L_\xi, \tag{19.10}$$

and

$$L_\xi = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \phi)^+ (D^\mu \phi) - V(\phi) - \frac{(\partial_\mu A^\mu + \xi g v \phi_2)^2}{2\xi}. \tag{19.11}$$

It remains to compute the determinant. When the gauge transformation is performed

$$A_\mu \to A_\mu + \partial_\mu \chi(x), \quad \phi_2 \to g\chi(x)\phi_1(x) \tag{19.12}$$

we find

$$G \to G + \partial^2 \chi(x) + \xi g^2 v \phi_1(x) \chi(x) - f(x), \tag{19.13}$$

so that

$$\det\left(\frac{\mathrm{d}G[A^\chi]}{\mathrm{d}\chi}\right) = \det\left(\partial^2 + \xi g^2 v\phi_1\right) = \det\left(\partial^2 + \xi g^2 v(v + h(x))\right). \quad (19.14)$$

This determinant *does not decouple* since it depends on the Higgs field. Hence, it must be written as an integral over ghost fields and added to the action. We find

$$\det\left(\partial^2 + \xi g^2 v(v + h(x))\right) = \int \mathcal{D}\bar{c}\mathcal{D}c \; e^{i\int \mathrm{d}^4x \bar{c}\left(-\partial^2 - \xi g^2 v(v+h(x))\right)c}. \quad (19.15)$$

The Lagrangian of the theory is given by L_ξ and the ghost Lagrangian that we can deduce from the above equation. We rewrite it using v, h and ϕ_2 fields. We find

$$\begin{aligned} L_{\text{full}} = &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\left(\partial_\mu h + gA_\mu\phi_2\right)^2 \\ &+ \frac{1}{2}\left(\partial_\mu\phi_2 - gA_\mu(v+h)\right)^2 - \frac{(\partial_\mu A^\mu + \xi g v\phi_2)^2}{2\xi} - V(\phi) \\ &- \bar{c}\left(\partial^2 + \xi g^2 v(v + h(x))\right)c. \end{aligned} \quad (19.16)$$

The important feature of this new Lagrangian is that the gauge fixing term also contains the cross terms $\partial_\mu A^\mu \phi_2$. When we combine the cross term from the kinetic energy of the ϕ field and from the gauge fixing term, we find the following contribution to the action

$$-gv\int \mathrm{d}^4x\left((\partial_\mu\phi_2)A_\mu + \phi_2\partial_\mu A^\mu\right) = -gv\int \mathrm{d}^4x\partial_\mu(\phi_2 A^\mu) = 0, \quad (19.17)$$

where we assume that all fields vanish at the infinitely remote three-dimensional sphere.

We can now investigate all the terms that are quadratic in the fields h, ϕ_2, A_μ and c to determine the mass spectrum of the theory and deduce propagators that we use to describe these fields. We find

$$\begin{aligned} &-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{g^2v^2}{2}A_\mu A^\mu \\ &+ \frac{1}{2}(\partial_\mu h)^2 - \frac{\lambda v^2}{4}h^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{\xi g^2 v^2\phi_2^2}{2} - \bar{c}\left(\partial^2 + \xi g^2 v^2\right)c. \end{aligned} \quad (19.18)$$

Hence, our theory has the gauge boson A^μ with the mass $m_A^2 = g^2 v^2$, the Higgs boson with the mass $\lambda v^2/2$ and would-be Goldstone field ϕ_2 and the ghost field c whose masses equal to $m_\phi^2 = \xi m_V^2$. We note that the unphysical nature of these particles is illuminated by the fact that their masses depend on an *arbitrary* gauge parameter ξ.

It is instructive to compute the propagator of the gauge field. Focusing on terms that are quadratic in A_μ, we write

$$S[A] = \int \mathrm{d}^4x \, \frac{1}{2} \, A^\mu \, T_{\mu\nu} \, A^\nu \tag{19.19}$$

where

$$T_{\mu\nu} = \left(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu\right) + \frac{1}{\xi}\partial_\mu\partial_\nu + m_V^2 g^{\mu\nu}. \tag{19.20}$$

When discussing the quantization of gauge fields we have shown that the propagator of the field A_μ is given by the inverse of $T_{\mu\nu}$

$$\langle 0 | T A_\mu(x_1) A_\nu(x_2) | 0 \rangle = i T^{-1}_{\mu\nu}(x_1, x_2). \tag{19.21}$$

To find the inverse of the operator $T^{\mu\nu}$, we switch to the momentum space and write

$$i T^{-1}_{\mu\nu}(x_1, x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} D_{\mu\nu}(k) e^{-i k_\alpha (x_1^\alpha - x_2^\alpha)}. \tag{19.22}$$

Then

$$T^{\mu\nu}(k) \; D_{\nu\rho}(k) = i g^\mu_\rho, \tag{19.23}$$

where

$$T_{\mu\nu}(k) = -k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu + m_v^2 g_{\mu nu}. \tag{19.24}$$

Computing the inverse, we find

$$D^{\mu\nu} = \frac{-i}{k^2 - m_v^2 + i0} \left(g^{\mu\nu} - (1 - \xi) \frac{k^\mu k^\nu}{k^2 - \xi m_V^2} \right). \tag{19.25}$$

The unitary (physical) gauge correspond to the formal limit $\xi \to \infty$. Note, however, that this limit is non-trivial; for example, the coupling between the ghost fields c and the Higgs boson h becomes very large in that limit.

We will continue with the discussion of the quantization of a non-Abelian gauge theory with the symmetry group $SU(2)$. The gauge symmetry is broken by the Higgs doublet. The Lagrangian reads

$$L = -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu} + (D^\mu \phi)^+ (D_\mu \phi) - V(|\phi|), \tag{19.26}$$

where

$$D_\mu \phi = \partial \phi - igT^a \phi. \tag{19.27}$$

Before we proceed, it is useful to simplify the kinetic term for the scalar field. We write

$$\begin{aligned}(D^\mu \phi)^+ (D_\mu \phi) = \partial^\mu \phi^+ \partial_\mu \phi + g^2 A^a_\mu A^{b,\mu} \phi^+ T^a T^b \phi \\ + igA^{a,\mu} \left(\phi^+ T^a \partial_\mu \phi - \partial_\mu \phi^+ T^a \phi \right).\end{aligned} \tag{19.28}$$

We are interested in the mixing terms that appear in the above equation once the electroweak symmetry breaking happens. In this case,

$$\phi = \phi_{\rm vac} + \tilde{\phi}, \tag{19.29}$$

where

$$\phi_{\rm vac} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{19.30}$$

and $\tilde{\phi}$ contains four fields that parameterize excitations around $\phi_{\rm vac}$.

To determine mixing terms, we need to extract from Eq. (**??**) terms that are quadratic in in fields, i.e. A_μ and $\tilde{\phi}$. To simplify this, we write $\tilde{\phi}$ as follows

$$\tilde{\phi} = \frac{h(x)}{v} \phi_{\rm vac} + \frac{2i}{v} \chi^a T^a \phi_{\rm vac}. \tag{19.31}$$

Writing generators T^a in terms of Pauli matrices, we obtain

$$\tilde{\phi} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_2 + i\chi_1 \\ h - i\chi_3 \end{pmatrix}, \tag{19.32}$$

which is indeed a complete parameterization.

We will now use the parameterization shown Eq. (19.31) to simplify the different terms in Eq. (19.28). First, it is easy to see that

$$\partial^\mu \phi^+ \partial_\mu \phi = \frac{1}{2} (\partial_\mu h)^2 + \frac{1}{2} \sum_{a=1}^{3} (\partial_\mu \chi_a)^2. \tag{19.33}$$

Then, since we are only interested in terms that are quadratic in the fields, we find

$$g^2 A^a_\mu A^{b,\mu}\phi^+ T^a T^b \phi = g^2 A^a_\mu A^{b,\mu}\phi^+_{\text{vac}} T^a T^b \phi_{\text{vac}} = \frac{1}{2}\frac{g^2 v^2}{4} A^a_\mu A^{a,\mu}, \qquad (19.34)$$

where we have used $T^a T^b + T^b T^a = \delta^{ab}/2$ which is valid for $SU(2)$ generators. This is the mass term for gauge bosons and we read off

$$m_A^2 = \frac{g^2 v^2}{4}. \qquad (19.35)$$

Finally, we consider the last term in Eq. (19.28). Since we are only interested in terms that are quadratic in the fields, we write

$$\begin{aligned}
&igA^{a,\mu}\left(\phi^+ T^a \partial_\mu \phi - \partial_\mu \phi^+ T^a \phi\right) \\
&\to igA^{a,\mu}\left(\phi^+_{\text{vac}} T^a \partial_\mu \tilde{\phi} - \partial_\mu \tilde{\phi}^+ T^a \phi_{\text{vac}}\right) \\
&\to -ig(\partial_\mu A^{a,\mu})\left(\phi^+_{\text{vac}} T^a \tilde{\phi} - \tilde{\phi}^+ T^a \phi_{\text{vac}}\right) \\
&= \frac{2g}{v}(\partial_\mu A^{a,\mu})\chi^b \phi^+_{\text{vac}}(T^a T^b + T^b T^a)\phi_{\text{vac}} = \frac{1}{2} g v (\partial_\mu A^{a,\mu})\chi^a.
\end{aligned} \qquad (19.36)$$

The mixing will have to be removed by the gauge-fixing term; if the gauge condition is chosen to be

$$G^a[A,\phi] = f^a(x), \qquad (19.37)$$

the change in the Lagrangian due to gauge fixing term becomes

$$L_{g.f.} = -\int \mathrm{d}^4 x \frac{(f^a(x))^2}{2\xi}. \qquad (19.38)$$

Hence, if we choose $G^a[A,\phi]$ as follows

$$G^a[A,\phi] = \partial_\mu A^{a,\mu} + \frac{1}{2}\xi g v \chi^a, \qquad (19.39)$$

we will ensure that the mixing terms cancel if the gauge fixing term is added to the original Lagrangian. When the above equation is substituted into gauge-fixing Lagrangian, the last term provides the mass to Higgs Goldstones. We find

$$m_\chi^2 = \xi \frac{g^2 v^2}{4} = \xi m_A^2. \qquad (19.40)$$

It remains to derive the functional determinant $\det(\delta G^a/\delta\theta^b)$ where θ^b are the parameters of the gauge transformation. This is obtained by performing the gauge transformations of the gauge field A^a_μ (which is identical to what we discussed when talking about non-Abelian gauge theories) and deriving the transformation of the fields χ^a from the representation of the field $\tilde{\phi}$ given in Eq. (19.31). Let us start with the χ fields. Starting with an infinitesimal gauge transformation

$$\phi \to U\phi, \tag{19.41}$$

where $U = e^{i\alpha_a T^a} \approx 1 + i\alpha_a T^a$, we obtain

$$\delta\phi = i\alpha_b T^b \left(\left(1 + \frac{h(x)}{v} \right) \phi_{\text{vac}} + \frac{2i}{v}\chi^a T^a \phi_{\text{vac}} \right). \tag{19.42}$$

After straightforward algebra, where we use the important property of $SU(2)$ generators

$$T^a T^b = \frac{1}{4}\delta^{ab} + \frac{1}{2} i\epsilon^{abc} T^c, \tag{19.43}$$

we find

$$\delta\phi = -\frac{1}{2v}\alpha_a \chi^a \phi_{\text{vac}} + \left(1 + \frac{h}{v}\right) i\alpha_a T^a \phi_{\text{vac}} - \frac{i}{v} T^a \epsilon_{abc} \alpha^b \chi^c \phi_{\text{vac}}. \tag{19.44}$$

It follows

$$\delta h(x) = -\frac{1}{2}\alpha^a \chi^a, \quad \delta\chi^a = \frac{1}{2}(v+h)\alpha^a - \frac{1}{2}\epsilon^{abc}\alpha^b \chi^c. \tag{19.45}$$

Since

$$\delta A^a_\mu = \frac{1}{g}\mathcal{D}_\mu \alpha^a, \tag{19.46}$$

we find

$$g\left(\frac{\delta G}{\delta \alpha}\right)^{ab} = (\partial_\mu \mathcal{D}^\mu)^{ab} + \frac{1}{2}\xi g^2 v \left(\frac{1}{2}(v+h)\delta^{ab} - \frac{1}{2}\epsilon^{abc}\chi_c \right). \tag{19.47}$$

Writing the determinant of this operator as an integral of ghosts fields, we observe that, similar to the Abelian case, the ghosts fields have the same masses as Higgs Goldstone bosons $m_c^2 = \xi g^2 v^2/4$, that they have standard couplings to the gauge fields (as dictated by the covariant derivative in the above formula) and that they interact with the Higgs field and the Higgs Goldstone fields χ^a.