TTP2
Lecture 20

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20 Chiral anomalies in the Standard Model

In lecture 14 we talked about the anomaly of the axial-vector singlet current. We derived an equation which related the divergence of the axial current with an operator that contains field-strength tensors of the electromagnetic field and argued that this result is exact. In this lecture we will discuss two things: 1) how to use this result to derive a decay rate of π_0 to two photons (a very important result in the history of QCD) and 2) why chiral anomalies are not a problem in the Standard Model in spite of the fact that we gauge the left- and the right-handed fermion fields differently.

We will start with the discussion of the decay of a neutral pion to two photons $\pi_0 \to \gamma\gamma$. To this end, consider a two-flavor QCD. The electromagnetic current reads

$$J_\mu^{\rm em} = \bar\psi \hat{Q} \gamma_\mu \psi, \tag{20.1}$$

where

$$\psi = \left(\begin{array}{c} u \\ d \end{array} \right), \tag{20.2}$$

and

$$\hat{Q} = \left(\begin{array}{cc} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{array} \right) = \frac{1}{6}\mathbf{1} + \frac{1}{2}\tau_3. \tag{20.3}$$

If we consider the correlator of an axial current

$$J_5^{a,\mu} = \bar\psi \frac{\tau^a}{2} \gamma^\mu \gamma_5 \psi \tag{20.4}$$

and two photons, the result will be proportional to

$${\rm Tr}\left[\tau^a \hat{Q}\hat{Q}\right] = {\rm Tr}\left[\tau^a \left(\frac{5}{18} + \frac{1}{6}\tau^3 \right)\right] = \frac{1}{3}\delta^{a3}. \tag{20.5}$$

We will also introduce the factor $N_c = 3$ which is the number of quark colors; we have to sum over quark colors when computing diagrams similar to those shown in Fig. 1 in two-flavor QCD. The divergence of the axial current becomes

$$\partial_\mu J_5^{a,\mu} = \frac{\alpha N_c \delta^{a3}}{12\pi} F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{20.6}$$

We will now show that this formula leads to an unambiguous prediction for the decay $\pi^0 \to \gamma\gamma$. Indeed, consider the matrix element $\langle \gamma(k_1)\gamma(k_2)| J_5^{a,\mu}|0\rangle$

and write its Lorentz decomposition that is consistent with Eq.(20.6). Then

$$\langle \gamma(k_1)\gamma(k_2)|J_5^{a,\mu}|0\rangle = \frac{iq_\mu}{q^2}\delta^{a3}\frac{\alpha N_c}{3\pi}\epsilon_{\alpha\beta\rho\sigma}\epsilon_1^\alpha\epsilon_2^\beta k_1^\rho k_2^\sigma \quad + \quad .... \tag{20.7}$$

where ellipses stand for contributions to the amplitude that are transversal w.r.t. q. The longitudinal term in Eq.(20.7) is written in such a way that it reproduces the anomaly equation Eq.(20.6) when it is contracted with q_μ. A very important aspect of Eq.(20.7) is that the matrix element there contains *a pole* in a variable q^2. Poles in kinematic variables appear in amplitudes for a reason – typically, they correspond to physical particles that appear in the spectrum of the theory.[1]

In our case, we assume that the $SU_L(2) \times SU_R(2)$ extended flavor symmetry is spontaneously broken and that the current $J_5^{a,\mu}$ is the current associated with the broken symmetry. In the broken phase, this current is written through a Goldstone boson field $J_{5\mu}^a = -\partial_\mu \pi^a$. Hence, the pole that we see in Eq.(20.7) is the contribution of a *neutral pion* and, given this relation between J_5^μ and the pion, we can write for the longitudinal part of the correlator

$$\langle \gamma(k_1)\gamma(k_2)|J_5^{a,\mu}|0\rangle = \frac{iF_\pi q^\mu}{q^2}A_{\pi_0\to\gamma\gamma} \tag{20.8}$$

By comparing Eq.(20.7) and Eq.(20.8), we obtain the amplitude for a decay of a pion to two photons

$$A_{\pi_0\to\gamma\gamma} = \frac{\alpha N_c}{3\pi F_\pi}\epsilon_{\alpha\beta\rho\sigma}\epsilon_1^\alpha\epsilon_2^\beta k_1^\rho k_2^\sigma. \tag{20.9}$$

It is worth emphasizing, again, that we were able to obtain the decay amplitude of a pion into two photon in a strongly-interacting theory. This amplitude is *exact* since anomalies do not receive higher-order corrections so that the prediction of a pole at $q^2 = 0$ is an unambiguous consequence of this fact.

We now change topic and turn to the discussion of anomalies in the Standard Model. This is an interesting problem because the Standard Model is a theory where left- and right fields are gauged differently. Hence, both vector and axial currents in the Standard Model are related to gauge fields and, therefore, they better not be anomalous. *This is indeed the case*. However, some work is needed to see how this conclusion comes about.

[1] Recall a similar logic in the discussion of Goldberger-Treiman relation.

Since anomaly manifests itself through an ultraviolet behavior of the theory, symmetry breaking of the Standard Model is largely irrelevant. We can therefore discuss three-point functions for $SU(2)_L \times U(1)_Y$ gauge fields, $W^{\pm}, W^3$, B. We should also consider gluon fields since they couple to electroweak fields through fermion loops. We will need the quantum numbers of leptons and quarks. We will take the first generation to write them down. We have

$$\hat{Y} \begin{pmatrix} \nu_l \\ e_L \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} \nu_l \\ e_L \end{pmatrix}, \quad \hat{Y} \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \frac{1}{6} \begin{pmatrix} u_l \\ d_L \end{pmatrix}, \tag{20.10}$$

$$\hat{Y} e_R = -e_R, \quad \hat{Y} u_R = \frac{2}{3} u_R, \quad \hat{Y} d_R = -\frac{1}{3} d_R. \tag{20.11}$$

Also, quarks have three colors and left- and right-fields contribute to the anomaly with opposite signs. The last feature arises because from three vertices in the triangle, one gets a product of three projection operators that turn into $1 \pm \gamma_5$ depending on wheter the projector is left or right. Since only γ_5 contributes to the anomaly left and right fermions indeed contribute with a different sign to anomalous correlators. In what follows, we will account for contributions of right quarks (leptons) with an additional minus sign relative to the contribution of left quarks (leptons).

The relevant correlators are then proportional to the following factors

- Bgg

$$\mathrm{Tr}\left[T^a T^b\right] \left[\left(\frac{1}{6}\right) \times 2 \,|_{u_L, d_L} + (-1) \left(\frac{2}{3}\right) |_{u_R} + (-1) \left(-\frac{1}{3}\right) |_{d_R} \right] = 0. \tag{20.12}$$

- BBB

$$\begin{aligned} & 2 \times \left(-\frac{1}{2}\right)^3 |_{e_L, \nu_L} + (-1)(-1)^3 \,|_{e_R} \\ & + N_c \left[2 \times \left(\frac{1}{6}\right)^3 |_{u_L, d_L} + (-1) \left(\frac{2}{3}\right)^3 |_{u_R} + (-1) \left(-\frac{1}{3}\right)^3 |_{d_R} \right] \\ = & -\frac{1}{4} + 1 + \frac{1}{36} - \frac{8}{9} + \frac{1}{9} = 0. \end{aligned} \tag{20.13}$$

- BW^3W^3

$$2\times\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)^2|_{e_L,\nu_L}+N_c 2\times\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^2|_{u_L,d_L}=-\frac{1}{4}+\frac{1}{4}=0. \tag{20.14}$$

- BW^+W^-

$$\left(\frac{1}{\sqrt{2}}\right)^2\left(\left(\frac{1}{2}\right)|_{e_L}+\left(\frac{1}{2}\right)|_{\nu_L}\right)+N_c\left(\frac{1}{\sqrt{2}}\right)^2\left(\left(-\frac{1}{6}\right)|_{u_L}+\left(-\frac{1}{6}\right)|_{d_L}\right)=0 \tag{20.15}$$

- W^3gg

$$\mathrm{Tr}\left[T^aT^b\right]\left(\frac{1}{2}|_{u_L}-\frac{1}{2}|_{d_L}\right)=0. \tag{20.16}$$

- $W^3W^3W^3$

$$\left(\frac{1}{2}\right)^3|_{\nu_L}-\left(\frac{1}{2}\right)^3|_{e_R}+N_c\left(\left(\frac{1}{2}\right)^3|_{u_L}-\left(\frac{1}{2}\right)^3|_{d_L}\right)=0. \tag{20.17}$$

It follows from the above equations that chiral anomalies cancel in the Standard Model. This happens separately for each of the three generations so anomaly cancellation does not provide any clue as to why there are three generations in Nature.