Problem set 1 for “Topology in condensed matter”

To be discussed in exercise class on November 7, 2023

1 Fundamental homotopy group π_1

This set of exercises will familiarize you with the basic terms. We will study the simplest homotopy group $\pi_1(\mathtt{S}^1)$, which defines topological properties of mappings $\mathtt{S}^1 \mapsto \mathtt{S}^1$. One of the ways to realize such mappings is to consider specific angular parametrization of a one-dimensional sphere both for the target space and for the source space, denoted as $\varphi(\theta)$. For the source space we can require $\theta \in [0, 2\pi]$; but since 0 and 2π define the same point on a circle, naively one would impose periodic boundary conditions $\varphi(0) = \varphi(2\pi)$. However, the correct requirement is that $\varphi(0)$ and $\varphi(2\pi)$ define the same point on the circle; and thus generally one should also allow mappings with $\varphi(2\pi) - \varphi(0) = 2\pi W$ with $W \in \mathbb{Z}$.

1. Consider two mappings:
$$\varphi_1(\theta) = \pi \sin\theta, \quad \varphi_2(\theta) = 4\pi \cos\theta \tag{1}$$
Show that they are *topologically equivalent* (*homotopic*, $\varphi_1 \sim \varphi_2$): i.e. provide explicitly a continuous map $H(\theta, t)$ (*homotopy*) such that $H(\theta, t = 0) = \varphi_1(\theta)$ and $H(\theta, t = 1) = \varphi_2(\theta)$. Show that both mappings are topologically equivalent to a trivial map $\varphi_0(\theta) = 0$.

2. A trivial map belongs to a wider family of linear maps:
$$\varphi_W(\theta) = W\theta \tag{2}$$
Show that they define a valid map only if $W \in \mathbb{Z}$. Prove that any pair of maps from this family are not topologically equivalent to each other; thus they can serve as the *representatives* of different *equivalence classes*. We define that any map that is homotopic to $\varphi_W(\theta)$ belongs to a class $[W]$; we've thus proven that mappings $\varphi_{1,2}(\theta)$ belong to a class $[0]$. In fact, it's not hard to prove that this classifies all the mappings.

3. Using the definition of a group operation from the lecture, show that map $\varphi_{W_1} * \varphi_{W_2}$ is homotopic to map $\varphi_{W_1+W_2}$. Show that it proves that (any map homotopic to φ_{W_1}) $*$ (any map homotopic to φ_{W_2}) is homotopic to $\varphi_{W_1+W_2}$. This statement can be shortly written as $[W_1] * [W_2] = [W_1 + W_2]$. We have just built an *isomorphism* between homotopy group $\pi_1(\mathtt{S}^1)$ and group of integer numbers $\mathbb{Z}$, i.e. proven that $\pi_1(\mathtt{S}^1) = \mathbb{Z}$.

2 Skyrmions

In this exercise you will study mappings $\mathtt{S}^2 \mapsto \mathtt{S}^2$, which define the homotopy group $\pi_2(\mathtt{S}^2)$. Such mappings can be realized as vector functions with a constraint:

$$\boldsymbol{n}(\boldsymbol{r}) = (n_x(\boldsymbol{r}), n_y(\boldsymbol{r}), n_z(\boldsymbol{r})), \quad \boldsymbol{n}^2(\boldsymbol{r}) = n_x^2(\boldsymbol{r}) + n_y^2(\boldsymbol{r}) + n_z^2(\boldsymbol{r}) = 1 \quad \Rightarrow \quad \boldsymbol{n}(\boldsymbol{r}) \in \mathtt{S}^2 \tag{3}$$

while for the source space it will be convenient to require:

$$\boldsymbol{r} = (x, y) \in \mathbb{R}^2, \quad \lim_{|\boldsymbol{r}|\to\infty} \boldsymbol{n}(\boldsymbol{r}) = \boldsymbol{n}_0, \tag{4}$$

because a two-dimensional plane where infinity is identified with a single point is topologically a two-dimensional sphere.

1. Prove that the following quantity is a topological invariant (called "topological charge"):

$$Q[\boldsymbol{n}] = \frac{1}{4\pi} \int d^2\boldsymbol{r}\, \boldsymbol{n}(\boldsymbol{r}) \cdot [\partial_x \boldsymbol{n}(\boldsymbol{r}) \times \partial_y \boldsymbol{n}(\boldsymbol{r})] \tag{5}$$

To do that, consider a continuous infinitesimal variation $\widetilde{\boldsymbol{n}}(\boldsymbol{r}) = \boldsymbol{n}(\boldsymbol{r}) + \epsilon\delta\boldsymbol{n}(\boldsymbol{r})$ (with $\epsilon \to 0$) and prove that $Q[\widetilde{\boldsymbol{n}}] = Q[\boldsymbol{n}]$ in the leading order in ϵ. Don't forget that variation should be consistent with $\widetilde{\boldsymbol{n}}^2(\boldsymbol{r}) = 1$!

2. Consider this quantity in the spherical coordinates for the target space $\boldsymbol{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ and in polar coordinates for the source space $\boldsymbol{r} = (r\cos\alpha, r\sin\alpha)$:

$$Q[\boldsymbol{n}] = \frac{1}{4\pi} \int_0^\infty dr \int_0^{2\pi} d\alpha\, \sin\theta\, (\partial_\alpha\theta\partial_r\phi - \partial_r\theta\partial_\alpha\phi) \tag{6}$$

Consider now for simplicity configurations with separable coordinates $\phi(r,\alpha) = \phi(\alpha)$ and $\theta(r,\alpha) = \theta(r)$. Note that such dependence is only possible if $\sin\theta(r \to 0) = \sin\theta(r \to \infty) = 0$. Show that topological charge is quantized, i.e. $Q[\boldsymbol{n}] \in \mathbb{Z}$.

3. Provide an an explicit formula for the configuration $\boldsymbol{n}(\boldsymbol{r})$ with an arbitrary given topological charge Q.

3 Berry curvature for spin s

This exercise will generalize result for the Berry curvature for spin-1/2 in a magnetic field derived in the lecture to an arbitrary spin s. The system is described by the following Hamiltonian:

$$\hat{H}(\boldsymbol{h}) = \boldsymbol{h} \cdot \hat{\boldsymbol{S}} \tag{7}$$

It has $2s+1$ eigenvalues $\epsilon_m = m|\boldsymbol{h}|$ corresponding to eigenstates denoted as $|m(\boldsymbol{h})\rangle$, with $m = -s, \ldots, s$.

On the lecture it was demonstrated that the Berry connection for a state with a given quantum number m can be obtained using the following formula:

$$\Omega_m^{\mu\nu}(\boldsymbol{h}) = i \sum_{m' \neq m} \left[\frac{\left\langle m(\boldsymbol{h}) \left| \frac{\partial \hat{H}(\boldsymbol{h})}{\partial h_\mu} \right| m'(\boldsymbol{h}) \right\rangle \left\langle m'(\boldsymbol{h}) \left| \frac{\partial \hat{H}}{\partial h_\nu} \right| m(\boldsymbol{h}) \right\rangle}{(\epsilon_m(\boldsymbol{h}) - \epsilon_{m'}(\boldsymbol{h}))^2} - c.c. \right], \tag{8}$$

which doesn't require calculation of derivatives of the instantaneous eigenfunctions $|m(\boldsymbol{h})\rangle$. Using this formula, show that the corresponding "Berry magnetic field" $b_m^\mu(\boldsymbol{h}) = \frac{1}{2}\epsilon^{\mu\nu\lambda}\Omega_m^{\nu\lambda}(\boldsymbol{h})$ corresponds to a magnetic field created by a "magnetic monopole" sitting at the origin $h = 0$:

$$b_m^\mu(\boldsymbol{h}) = -m\frac{h^\mu}{|\boldsymbol{h}|^3} \tag{9}$$

Hint: One can greatly simplify the calculation by noting that $\Omega_m^{\mu\nu}$ transforms under 3D rotations as a second-rank tensor, thus making it possible to perform a rotation to coordinate system where $\boldsymbol{h}$ aligned parallel to z-axis, $\boldsymbol{h} = (0, 0, |\boldsymbol{h}|)$. Show that only non-zero components of Ω are Ω^{xy} and Ω^{yx}, and calculate them using the known matrix elements of the ladder operators $\hat{S}^\pm \equiv \hat{S}^x \pm i\hat{S}^y$ for the quantum spin algebra:

$$\hat{S}^+ |m\rangle = \sqrt{s(s+1) - m(m+1)}\, |m+1\rangle\,, \quad \hat{S}^- |m\rangle = \sqrt{s(s+1) - m(m-1)}\, |m-1\rangle \tag{10}$$

What is the corresponding "Berry magnetic field"? Use symmetry considerations to generalize obtained result for an arbitrary direction of $\boldsymbol{h}$.

4 Berry connection in the degenerate case

The adiabatic theorem states that if the evolution is sufficiently slow, the transitions between states with different energies are negligible. If, however, the instantaneous Hamiltonian contains degenerate subspaces, the transitions between them are allowed, and the corresponding adiabatic evolution might be non-trivial. In this exercise you will explore the implications of this.

1. Consider arbitrary degenerate subspace with degeneracy d, spanned by eigenvectors $\{|a(\boldsymbol{R})\rangle\}_{a=1}^{d}$. At any time moment of the adiabatic evolution, the wavefunction can be expanded as the linear superposition of those vectors:

$$|\psi(t)\rangle = \sum_a c_a(t)\,|a(\boldsymbol{R}(t))\rangle \tag{11}$$

   Show that the expansion coefficients obey system of linear equations of motion:

$$\frac{dc_a(t)}{dt} = i\sum_{\alpha,b} \frac{dR^\alpha}{dt} A^\alpha_{ab}(\boldsymbol{R}(t))c_b(t) \tag{12}$$

   and derive the generic expression for the *non-Abelian Berry connection* $A^\alpha_{ab}(\boldsymbol{R})$, which in this case becomes a $d \times d$ matrix.
   Note: unlike the non-degenerate case where this equation can be solved explicitly and gives rise to the Berry phase

$$c_a(t) = \exp(i\gamma_a(t)), \quad \gamma_a(t) = \int_{\boldsymbol{R}(0)}^{\boldsymbol{R}(t)} A(\boldsymbol{R})d\boldsymbol{R}, \tag{13}$$

   for the degenerate case the adiabatic evolution can be highly non-trivial. This is because the generic matrix $A^\alpha_{ab}(\boldsymbol{R})$ might not commute with itself for different values of $\boldsymbol{R}$, justifying the name "non-Abelian".

2. Now consider a specific example. A free particle in a two-dimensional infinite deep potential well of size $L \times L$:

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + U_0(x,y), \quad U_0(x,y) = \begin{cases} 0, & |x| < \frac{L}{2} \quad \& \quad |y| < \frac{L}{2} \\ \infty, & \text{otherwise} \end{cases} \tag{14}$$

   All the eigenstates of such system can be found in a separable form and expressed via eigenfunctions of the one-dimensional infinite deep potential well $\psi_n(x) = \sqrt{2/L}\cdot\sin\left(\pi n\left[x/L - 1/2\right]\right)$ as follows:

$$\psi_{nm}(x,y) = \psi_n(x)\psi_m(y), \quad (n,m = 1,2,3,\dots) \tag{15}$$

   What are the corresponding eigenvalues ϵ_{nm}? Show that $|\psi_{12}\rangle$ and $|\psi_{21}\rangle$ form a twofold degenerate subspace.

3. The well is adiabatically rotated along z-axis, so that the Hamiltonian acquires explicit dependence on the rotation angle φ:

$$\hat{H}(\varphi) = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + U_0(\widetilde{x},\widetilde{y}), \quad \begin{cases} \widetilde{x} = x\cos\varphi + y\sin\varphi \\ \widetilde{y} = -x\sin\varphi + y\cos\varphi \end{cases} \tag{16}$$

   Calculate explicitly the corresponding Berry connection $A_{ab}(\varphi)$.

4. The system was initially prepared in the state $|\psi_{12}\rangle$. Find the state of the system after the box is rotated by $\varphi = 2\pi$. What is the probability to remain in the same state after the full evolution?