Solutions to problem set 1 for “Topology in condensed matter”

Discussed in exercise class on November 7, 2023

1 Fundamental homotopy group π_1

1. The homotopy between two mappings φ_1 and φ_2 belonging to the same equivalence class $[W]$ (i.e. $\varphi_i(2\pi) = 2\pi W$) is trivial:
$$H_{\varphi_1 \mapsto \varphi_2}(\theta, t) = \varphi_1(\theta) + t \cdot (\varphi_2(\theta) - \varphi_1(\theta)) \tag{1}$$
At each t it gives proper mapping belonging to the same equivalence class, i.e. $H(2\pi, t) = 2\pi W$. With this construction we can prove $\varphi_1 \sim \varphi_2 \sim \varphi_0$.

2. The fact that linear map $\varphi_W(\theta) = W\theta$ defines a proper map only for integer W is trivial, as previously we have required that $(\varphi(2\pi) - \varphi(0))/2\pi \equiv W \in \mathbb{Z}$.

3. The product of two linear maps is defined as follows (generalizing the definition from the lecture):
$$(\varphi_{W_1} * \varphi_{W_2})(\theta) = \begin{cases} 2W_1\theta, & \theta \in [0, \pi] \\ 2W_2\theta + 2(W_1 - W_2)\pi, & \theta \in [\pi, 2\pi] \end{cases}, \tag{2}$$
At $\theta = 2\pi$ it gives $2\pi(W_1 + W_2)$. Thus we can again use the construction from Eq. (1) to prove homotopy $\varphi_{W_1} * \varphi_{W_2} \sim \varphi_{W_1+W_2}$.

4. Let $\varphi_i \sim \varphi_{W_i}$ (with $i = 1, 2, 3$ and $W_3 \equiv W_1 + W_2$), i.e. there exists three homotopies $H_{\varphi_i \mapsto \varphi_{W_i}}(\theta, t)$. Then we can build explicitly a homotopy:
$$H_{\varphi_1 * \varphi_2 \mapsto \varphi_3}(\theta, t) = \begin{cases} H_{\varphi_1 \mapsto \varphi_{W_1}}(\theta, 3t) * H_{\varphi_2 \mapsto \varphi_{W_2}}(\theta, 3t) & t \le 1/3 \\ H_{\varphi_{W_1} * \varphi_{W_2} \mapsto \varphi_{W_3}}(\theta, 3t - 1), & 1/3 < t < 2/3 \\ H_{\varphi_3 \mapsto \varphi_{W_3}}(\theta, 3 - 3t), & 2/3 < t \le 1 \end{cases} \tag{3}$$
This proves $[W_1] * [W_2] = [W_1 + W_2]$, i.e. $\pi_1(\mathtt{S}^1) = \mathbb{Z}$.

2 Skyrmions

1. Variation of the topological term reads (in the second line we integrated second and third term by parts):
$$\begin{aligned} \delta Q/\epsilon = \frac{1}{4\pi} \int d^2\boldsymbol{r}\, (\delta \boldsymbol{n}(\boldsymbol{r}) \cdot [\partial_x \boldsymbol{n}(\boldsymbol{r}) \times \partial_y \boldsymbol{n}(\boldsymbol{r})] + \partial_x \delta \boldsymbol{n}(\boldsymbol{r}) \cdot [\partial_y \boldsymbol{n}(\boldsymbol{r}) \times \boldsymbol{n}(\boldsymbol{r})] + \partial_y \delta \boldsymbol{n}(\boldsymbol{r}) \cdot [\boldsymbol{n}(\boldsymbol{r}) \times \partial_x \boldsymbol{n}(\boldsymbol{r})]) \\ = \frac{3}{4\pi} \int d^2\boldsymbol{r}\, \delta \boldsymbol{n}(\boldsymbol{r}) \cdot [\partial_x \boldsymbol{n}(\boldsymbol{r}) \times \partial_y \boldsymbol{n}(\boldsymbol{r})] \end{aligned} \tag{4}$$
But since $\partial_x \boldsymbol{n}(\boldsymbol{r}) \perp \boldsymbol{n}(\boldsymbol{r})$ and $\partial_y \boldsymbol{n}(\boldsymbol{r}) \perp \boldsymbol{n}(\boldsymbol{r})$ (and $\boldsymbol{n}(\boldsymbol{r})$ is three-component vector), their vector product should be collinear with $\boldsymbol{n}(\boldsymbol{r})$. However, $\delta \boldsymbol{n}(\boldsymbol{r}) \perp \boldsymbol{n}(\boldsymbol{r})$, and thus this variation is zero.

2. In the spherical coordinates the topological charge reads:
$$Q[\boldsymbol{n}] = \frac{1}{4\pi} \int d^2\boldsymbol{r} \sin\theta (\partial_y \varphi \partial_x \theta - \partial_y \theta \partial_x \varphi) \tag{5}$$

and in polar coordinates this transforms to:

$$\partial_x = \frac{\partial r}{\partial x}\partial_r + \frac{\partial \alpha}{\partial x}\partial_\alpha = \cos\alpha\partial_r + \frac{\sin\alpha}{r}\partial_\alpha, \quad \partial_y = \sin\alpha\partial_r - \frac{\cos\alpha}{r}\partial_\alpha \tag{6}$$

and thus:

$$Q[\boldsymbol{n}] = \frac{1}{4\pi}\int_0^\infty dr \int_0^{2\pi} d\alpha\ (\partial_\alpha\theta\partial_r\phi - \partial_r\theta\partial_\alpha\phi)\sin\theta \tag{7}$$

For separable coordinate dependence one has:

$$Q[\boldsymbol{n}] = \frac{[\phi(2\pi) - \phi(0)]\,[\cos\theta(\infty) - \cos\theta(0)]}{4\pi}$$

Finally, we have $\varphi(2\pi) - \varphi(0) = 2\pi W$ with $W \in \mathbb{Z}$ and $\theta(\infty), \theta(0) \in 0, \pi$, thus $\cos(\theta(\infty)) - \cos(\theta(0)) = 0, \pm 2$ — which gives $Q[\boldsymbol{n}] \in \pm W, 0$. Since W can be arbitrary integer number, this shows quantization of Q.

3. Explicit example can be built as follows:

$$\cos\theta(r) = \tanh r, \quad \varphi(\alpha) = Q\alpha \tag{8}$$

and thus

$$\begin{cases} n_x = \sin\theta\cos\varphi & = \frac{\cos(Q\arctan\frac{y}{x})}{\cosh r} \\ n_y = \sin\theta\sin\varphi & = \frac{\sin(Q\arctan\frac{y}{x})}{\cosh r} \\ n_z = \cos\theta & = \tanh r \end{cases} \tag{9}$$

3 Berry curvatore for spin s

We start with the formula given in the task, noting that $\partial\hat{H}(\boldsymbol{h})/\partial h_\mu = \hat{S}^\mu$:

$$\Omega_m^{\mu\nu}(\boldsymbol{h}) = \frac{i}{|h|^2}\sum_{m'\neq m}\left[\frac{\left\langle m(\boldsymbol{h})\middle|\hat{S}_\mu\middle|m'(\boldsymbol{h})\right\rangle\left\langle m'(\boldsymbol{h})\middle|\hat{S}_\nu\middle|m(\boldsymbol{h})\right\rangle}{(m-m')^2} - c.c.\right],$$

Let's perform a rotation (acting on μ, ν indices) such that quantization axis coincides with z-axis. In such coordinate frame, the only contribution comes from $m' = m \pm 1$, and we can utilize:

$$\hat{S}_+\left|m\right\rangle = \sqrt{s(s+1) - m(m+1)}\left|m+1\right\rangle, \quad \hat{S}_-\left|m\right\rangle = \sqrt{s(s+1) - m(m-1)}\left|m-1\right\rangle \tag{10}$$

Thus:

$$\begin{aligned}\Omega_m^{xy}(\boldsymbol{h}) &= \frac{i}{|h|^2}\sum_{m'=m\pm1}\left(\left\langle m\middle|\hat{S}_x\middle|m'\right\rangle\left\langle m'\middle|\hat{S}_y\middle|m\right\rangle - c.c.\right) \\ &= \frac{i}{|h|^2}\left(\frac{1}{4i}\left[\left\langle m\middle|\hat{S}_-\middle|m+1\right\rangle\left\langle m+1\middle|\hat{S}_+\middle|m\right\rangle - \left\langle m\middle|\hat{S}_+\middle|m-1\right\rangle\left\langle m-1\middle|\hat{S}_-\middle|m\right\rangle\right] - c.c.\right) = -\frac{m}{|h|^2}\end{aligned} \tag{11}$$

We also have $\Omega_m^{yx} = -\Omega_m^{xy}$, thus $b_m^z = -m/|h|^2$. Performing coordinate transformation back to the reference frame, we obtain

$$\boldsymbol{b}(\boldsymbol{h}) = -m\boldsymbol{h}/|h|^3 \tag{12}$$

4 Berry connection in the degenerate case

1. Let's substitute this expansion to the Schrodinger equation:

$$i\partial_t\left|\psi\right\rangle - \hat{H}\left|\psi\right\rangle = \exp\left(-i\int^t E(\boldsymbol{R}(\tau))d\tau\right)\sum_a\left(i\frac{dc_a}{dt}\left|a(\boldsymbol{R}(t))\right\rangle + c_a(t)\frac{\partial}{\partial\boldsymbol{R}}\left|a(\boldsymbol{R}(t))\right\rangle\frac{d\boldsymbol{R}}{dt}\right) = 0 \tag{13}$$

Projecting it onto $\langle b(\boldsymbol{R}(t))|$, we obtain:

$$\frac{dc_a}{dt} + \sum_b c_b(t) \langle a(\boldsymbol{R}(t))| \frac{\partial}{\partial \boldsymbol{R}} |b(\boldsymbol{R}(t))\rangle \frac{d\boldsymbol{R}}{dt} = 0 \Leftrightarrow \frac{dc_a}{dt} = i \sum_b \frac{d\boldsymbol{R}}{dt} \boldsymbol{A}_{ab}(\boldsymbol{R}(t)) c_b(t) = 0, \tag{14}$$

with $A_{ab}(\boldsymbol{R}) = i \langle a(\boldsymbol{R})| \frac{\partial}{\partial \boldsymbol{R}} |b(\boldsymbol{R})\rangle$.

2. For the particle in a box, eigenenergies are $\epsilon_{nm} = \pi^2(n^2+m^2)/2ML^2$. The degeneracy between (n,m) and (m,n) is thus obvious.

3. It's easy to see that diagonal elements of the Berry connection are zero, while offdiagonal are constant and are given by:

$$A_{12} = -\int_0^L dx \int_0^L dy \cdot \frac{2}{L} \sin\frac{\pi x}{L} \sin\frac{2\pi x}{L} \cdot [-ix\partial_y + iy\partial_x] \frac{2}{L} \sin\frac{2\pi x}{L} \sin\frac{\pi y}{L} = \frac{256i}{27\pi^2} = A_{21}^* \tag{15}$$

4. The equations for the adiabatic evolution are:

$$\begin{cases} c_1'(\varphi) = iA_{12}c_2(\varphi) \\ c_2'(\varphi) = iA_{21}c_1(\varphi) \end{cases} \Rightarrow \begin{cases} c_1(\varphi) = c_1(0)\cos A\varphi - c_2(0)\sin A\varphi \\ c_2(\varphi) = c_1(0)\sin A\varphi + c_2(0)\cos A\varphi \end{cases}, \quad A = -iA_{12} = \frac{256}{27\pi^2} \tag{16}$$

thus:

$$|\psi_f\rangle = \cos\frac{512}{27\pi} |\psi_{12}\rangle + \sin\frac{512}{27\pi} |\psi_{21}\rangle \tag{17}$$

Probability to remain in the same state is $P = \cos^2 \frac{512}{27\pi} \approx 0.94$.