Solutions to problem set 2 for "Topology in condensed matter"

Discussed in exercise class on November 21, 2023

1 Semiclassical description of quantum spin

Without loss of generality, we can consider magnetic field parallel to z axis: $\boldsymbol{h} = (0, 0, h)$. The action then reads:

$$A[\varphi, \theta] = S \int dt \left[-\dot{\varphi}(1 - \cos\theta) - h\cos\theta \right] \tag{1}$$

The semiclassical equations of motion are thus:

$$\frac{\delta A}{\delta\theta(t)} = \frac{\partial L}{\partial\theta} = S\sin\theta\,(-\dot{\varphi} + h) = 0 \Rightarrow \dot{\varphi} = h \tag{2}$$

$$\frac{\delta A}{\delta\varphi(t)} = -\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}} = \frac{d}{dt}\cos\theta = 0 \Rightarrow \theta = \text{const} \tag{3}$$

which exactly corresponds to the following equations of motion for spin components:

$$\begin{cases} \dot{S}_x = -hS_y \\ \dot{S}_y = hS_x \\ \dot{S}_z = 0 \end{cases} \Rightarrow \frac{d\boldsymbol{S}}{dt} = [\boldsymbol{h} \times \boldsymbol{S}(t)] \tag{4}$$

Note that in the Heisenberg picture this is just equations of motion for the spin operator:

$$\frac{d\hat{S}_\alpha}{dt} = i\left[\hat{H}, \hat{S}_\alpha\right] = i\left[h_\beta\hat{S}_\beta, \hat{S}_\alpha\right] = -\epsilon_{\beta\alpha\gamma}h_\beta S_\gamma = [\boldsymbol{h} \times \boldsymbol{S}]_\alpha \tag{5}$$

2 Jackiw-Rebbi model

1. Denote $k_x = k\cos\phi$ and $k_y = k\sin\phi$; then the Hamiltonian reads:

$$\hat{H} = \begin{pmatrix} \Delta & v_F k e^{-i\phi} \\ v_F k e^{i\phi} & -\Delta \end{pmatrix}. \tag{6}$$

Denote also $E = \sqrt{v_F^2 k^2 + \Delta^2}$. Then its normalized eigenvectors are:

$$E_+ = E, \quad |\boldsymbol{k}, +\rangle = \frac{1}{\sqrt{2E(E+\Delta)}} \begin{pmatrix} e^{-i\phi}(E + \Delta) \\ v_F k \end{pmatrix}, \tag{7}$$

$$E_- = -E, \quad |\boldsymbol{k}, -\rangle = \frac{1}{\sqrt{2E(E-\Delta)}} \begin{pmatrix} e^{-i\phi}(E - \Delta) \\ -v_F k \end{pmatrix}. \tag{8}$$

2. The given formula for the Berry curvature for both bands yields:

$$\Omega_+(\boldsymbol{k}) = -2\text{Im}\frac{\langle \boldsymbol{k}, +|\, v_F\hat{\sigma}_x\, |\boldsymbol{k}, -\rangle \langle \boldsymbol{k}, -|\, v_F\hat{\sigma}_y\, |\boldsymbol{k}, +\rangle}{4E^2} = -\frac{v_F^2\Delta}{2E^3} \tag{9}$$

$$\Omega_-(\boldsymbol{k}) = -2\text{Im}\frac{\langle \boldsymbol{k}, -|\, v_F\hat{\sigma}_x\, |\boldsymbol{k}, +\rangle \langle \boldsymbol{k}, +|\, v_F\hat{\sigma}_y\, |\boldsymbol{k}, -\rangle}{4E^2} = \frac{v_F^2\Delta}{2E^3}, \tag{10}$$

and thus the Chern numbers read:

$$C_+ = -\frac{1}{2\pi}\int_0^\infty 2\pi k dk \frac{v_F^2\Delta}{2(v_F^2k^2 + \Delta^2)^{3/2}} = -\frac{\Delta}{4}\int_0^\infty \frac{dz}{(z + \Delta^2)^{3/2}} = -\frac{\text{sign}\Delta}{2}, \quad C_- = -C_+. \tag{11}$$

3. In the Fourier domain, the given lattice model corresponds to the same Hamiltonian up to replacement $k_x \mapsto \sin(k_x a)/a$ and $k_y \mapsto \sin(k_y a)/a$:

$$\hat{H}'(\boldsymbol{k}) = \frac{v_F}{a}\left(\hat{\sigma}_x \sin k_x a + \hat{\sigma}_y \sin k_y a\right) + \Delta\hat{\sigma}_z, \tag{12}$$

and, clearly, its Taylor expansion in $k_{x,y}a \ll 1$ reproduces the original Dirac Hamiltonian.

4. The expansion is straightforward:

$$\boldsymbol{K}_1 = (0, \pi/a) \Rightarrow \hat{H}_1(\boldsymbol{p}) = \hat{H}'(\boldsymbol{K}_1 + \boldsymbol{p}) \approx v_F(\hat{\sigma}_x p_x - \hat{\sigma}_y p_y) + \Delta\hat{\sigma}_z, \tag{13}$$

$$\boldsymbol{K}_2 = (\pi/a, 0) \Rightarrow \hat{H}_2(\boldsymbol{p}) = \hat{H}'(\boldsymbol{K}_2 + \boldsymbol{p}) \approx v_F(-\hat{\sigma}_x p_x + \hat{\sigma}_y p_y) + \Delta\hat{\sigma}_z, \tag{14}$$

$$\boldsymbol{K}_3 = (\pi/a, \pi/a) \Rightarrow \hat{H}_3(\boldsymbol{p}) = \hat{H}'(\boldsymbol{K}_3 + \boldsymbol{p}) \approx v_F(-\hat{\sigma}_x p_x - \hat{\sigma}_y p_y) + \Delta\hat{\sigma}_z. \tag{15}$$

Each Hamiltonian indeed describes own Dirac cone.

5. So far we have obtained Hamiltonians of the following form:

$$\hat{H} = v_F\left(s_1\hat{\sigma}_x p_x + s_2\hat{\sigma}_y p_y\right) + \Delta\hat{\sigma}_z, \quad s_{1,2} = \pm 1 \tag{16}$$

Then

$$\Omega_\pm(\boldsymbol{p}) = -2\mathrm{Im}\frac{\langle \boldsymbol{p}, \pm|\, s_1 v_F \hat{\sigma}_x \,|\boldsymbol{p}, \mp\rangle \langle \boldsymbol{p}, \mp|\, s_2 v_F \hat{\sigma}_y \,|\boldsymbol{p}, \pm\rangle}{4E^2} \propto s_1 s_2 \tag{17}$$

However, as each sign change corresponds to replacement $\phi \mapsto -\phi$ (for H_1), $\phi \mapsto \pi - \phi$ (for H_2) and $\phi \mapsto \pi + \phi$ (for H_3), while the combination of matrix elements is ϕ-independent. Thus:

$$\Omega_-^{(1)}(\boldsymbol{p}) = \Omega_-^{(2)}(\boldsymbol{p}) = -\Omega_-^{(3)}(\boldsymbol{p}) = -\Omega_-^{(0)}(\boldsymbol{p}), \tag{18}$$

and the total Chern number of the lower band vanishes:

$$C_- = C_-^{(0)} + C_-^{(1)} + C_-^{(2)} + C_-^{(3)} = \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}\right)\mathrm{sign}\Delta = 0 \tag{19}$$

Since $C_+ + C_- = 0$, it also implies vanishing of the Chern number for the upper band.

6. Now we switch back to the inhomogeneous case. Substituting the proposed ansatz to the Schroediner equation, we obtain:

$$\begin{cases} (c_1 + ic_2)\Delta(x) - ic_2 v_F k_y = Ec_1 \\ i(c_1 + ic_2)\Delta(x) + ic_1 v_F k_y = Ec_2 \end{cases} \tag{20}$$

which indeed has a solution provided $c_1 + ic_2 = 0$ and $E = v_F k_y$. This solution describes a chiral (because the group velocity is $\partial E/\partial k_y = v_F > 0$, the mode propagates in a single direction) edge modes localized at the boundary between two topologically distinct bulk phases.

3 Lattice models in magnetic field: Chern insulators

1. Explicit calculation gives:

$$\begin{aligned}\hat{T}_x^{-1}\hat{H}\hat{T}_x &= -\hat{T}_x^{-1}\sum_{m,n}\Big(t_x e^{-2\pi i n\Phi/\Phi_0}\,|m,n\rangle\langle m+2,n| + t_y e^{-2\pi i(m+n)\Phi/\Phi_0}\,|m,n-1\rangle\langle m+1,n|\Big) + h.c. \\ &= -\sum_{m,n}\Big(t_x\,|m+1,n\rangle\langle m+2,n| + t_y e^{-2\pi i(m+1)\Phi/\Phi_0}\,|m+1,n-1\rangle\langle m+1,n|\Big) + h.c. = \hat{H}\end{aligned} \tag{21}$$

Commutation with $\hat{T}_y$ is trivial because Hamiltonian is explicitly translationally invariant in y direction:

$$\hat{T}_y^{-1}\hat{H}\hat{T}_y = \hat{H} \tag{22}$$

which proves $[\hat{H}, \hat{T}_x] = [\hat{H}, \hat{T}_y] = 0$. Finally, one has:

$$\hat{T}_y^{-1}\hat{T}_x^q\hat{T}_y = \hat{T}_y^{-1}\sum_{m,n} e^{-2\pi i q n\Phi/\Phi_0}\,|m-q,n\rangle\langle m,n+1| = e^{2\pi i q\Phi/\Phi_0}\hat{T}_x^q \tag{23}$$

which proves that for $\Phi/\Phi_0 = p/q$ with $p, q \in \mathbb{Z}$, one has $[\hat{T}_x^q, \hat{T}_y] = 0$.

2. Direct substitution of the Ansatz to the Schroedinger equation yields:

$$\hat{H}\left|k_x,k_y\right\rangle = -\sum_{a=1}^{q}\sum_{m,n} e^{iqk_xm+ik_yn}\left|a+mq,n\right\rangle\left(t_x(\psi_{a+1}+\psi_{a-1})+2t_y\cos(k_y-2\pi a\Phi/\Phi_0)\psi_a\right)$$

   Careful analysis of terms with $a=1$ and $a=q$ show that in this expression instead of amplitude ψ_0 one should have $\psi_q e^{-iqk_x}$, while instead of amplitude ψ_{q+1} one should have $\psi_1 e^{iqk_x}$; this precisely reproduces twisted periodic boundary conditions obtained earlier. Finally, comparing it with the RHS of the Schroedinger equation:

$$E(k_x,k_y)\left|k_x,k_y\right\rangle = \sum_{a=1}^{q}\sum_{m,n} e^{iqk_xm+ik_yn}\left|a+mq,n\right\rangle E(k_x,k_y)\psi_a \tag{24}$$

   we recover the Harper equation:

$$-\left(t_x(\psi_{a+1}+\psi_{a-1})+2t_y\cos(k_y-2\pi a\Phi/\Phi_0)\psi_a\right) = E(k_x,k_y)\psi_a \tag{25}$$

3. At $t_x=0$, the intersection happens at:

$$E_1^{(0)}(k_y)=E_2^{(0)}(k_y)\Rightarrow k_y=\{0,\pi\} \tag{26}$$

$$E_1^{(0)}(k_y)=E_3^{(0)}(k_y)\Rightarrow k_y=\{-\pi/3,2\pi/3\} \tag{27}$$

$$E_2^{(0)}(k_y)=E_3^{(0)}(k_y)\Rightarrow k_y=\{-2\pi/3,\pi/3\} \tag{28}$$

4. Let's focus on intersection around $k_y=-\pi/3+p_y$:

$$\hat{H}_{13}(k_x,p_y) = -t_y - \begin{pmatrix} -\sqrt{3}t_yp_y & t_xe^{-3ik_x} \\ t_xe^{3ik_x} & \sqrt{3}t_yp_y \end{pmatrix} \tag{29}$$

   This Hamiltonian has two eigenvectors, for the lowest and middle band; denote $E=\sqrt{t_x^2+3t_y^2p_y^2}$

$$E_- = -t_y - E, \quad \left|-\right\rangle = \frac{1}{\sqrt{2E(E-\sqrt{3}t_yp_y)}}\begin{pmatrix} e^{-3ik_x}(E-\sqrt{3}t_yp_y) \\ t_x \end{pmatrix} \tag{30}$$

$$E_+ = -t_y + E, \quad \left|+\right\rangle = \frac{1}{\sqrt{2E(E+\sqrt{3}t_yp_y)}}\begin{pmatrix} e^{-3ik_x}(E+\sqrt{3}t_yp_y) \\ -t_x \end{pmatrix} \tag{31}$$

   Then the Berry curvature for the lowest band yields:

$$\Omega_-(\boldsymbol{k}) = -2\mathrm{Im}\frac{\left\langle -\right|\partial\hat{H}_{13}/\partial k_x\left|+\right\rangle\left\langle +\right|\partial\hat{H}_{13}/\partial p_y\left|-\right\rangle}{(E_+-E_-)^2} = \frac{3\sqrt{3}t_x^2t_y}{2E^3} \tag{32}$$

   and the Chern number gives:

$$C_- = \frac{1}{2\pi}\int_{-\pi/3}^{\pi/3}dk_x\int_{-\infty}^{\infty}dp_y\cdot\frac{3\sqrt{3}t_x^2t_y}{2(t_x^2+3t_y^2p_y^2)^{3/2}} = +1 \tag{33}$$

   The Chern numbers of all bands are thus:

$$C_{\text{upper}} = C_{\text{lower}} = +1, \quad C_{\text{middle}} = -2 \tag{34}$$