# Solutions to problem set 4 for "Topology in condensed matter"

Discussed in exercise class on December 19, 2023

## 1 Semiclassical description of Landau levels in graphene

1. The first equation of motion gives

$$\frac{d\boldsymbol{R}}{dt} = \frac{\partial E}{\partial \boldsymbol{P}} = v_G^2 \frac{\boldsymbol{P}}{E} \quad \Rightarrow \quad \frac{d\boldsymbol{P}}{dt} = \frac{ev_G^2 B}{cE} [\boldsymbol{P} \times \hat{\boldsymbol{z}}] \tag{1}$$

with $\hat{\boldsymbol{z}}$ being unit vector parallel to $z$-axis. Within angular parametrization, we obtain:

$$\begin{cases} \frac{dP_x}{dt} = \frac{ev_G^2 B}{cE} P_y \\ \frac{dP_y}{dt} = -\frac{ev_G^2 B}{cE} P_x \end{cases} \Leftrightarrow \frac{d\phi}{dt} = \omega_c = -\frac{ev_G^2 B}{cE} \Rightarrow \phi(t) = \phi_0 + \omega_c t \tag{2}$$

(note that since $e < 0$, the cyclotron frequency $\omega_c = |e|v_G^2 B/cE$ is positive). Integrating these equations, we obtain for the coordinate dependence:

$$\begin{cases} X(t) = X_0 + R\sin(\phi_0 + \omega_c t) \\ Y(t) = Y_0 - R\cos(\phi_0 + \omega_c t) \end{cases} , \quad R \equiv \frac{v_G^2 P}{\omega_c E} = \frac{cE}{|e|v_G B} \tag{3}$$

with $(X_0, Y_0)$ being arbitrary integration constants denoting the center of the cyclotron orbit.

2. The semiclassical action calculated along this cyclotron orbit then reads:

$$S_1 = \oint \boldsymbol{P} d\boldsymbol{R} = \int_0^{2\pi/\omega_c} \boldsymbol{P} \frac{d\boldsymbol{R}}{dt} dt = \frac{2\pi E}{\omega_c} = \frac{2\pi c E^2}{|e|v_G^2 B} \tag{4}$$

$$S_2 = \frac{e}{c} \oint \boldsymbol{A} d\boldsymbol{R} = \frac{e}{c} B \cdot \pi R^2 = -\frac{\pi c E^2}{|e|v_G^2 B} \tag{5}$$

and the quantization is:

$$E_n = \sqrt{eBv_G^2(2n+1)/c} \tag{6}$$

3. The Berry phase was calculated earlier in the problem set 2, and reads:

$$S_3 \equiv \oint \widetilde{\boldsymbol{A}}(\boldsymbol{P}) d\boldsymbol{P} \approx \lim_{\Delta \to 0} \iint_{\mathbb{R}^2} \Omega_z(\boldsymbol{P}) d^2\boldsymbol{P} = -\frac{v_F^2 \Delta}{2} \int_0^\infty \frac{2\pi P dP}{(v_G^2 P^2 + \Delta^2)^{3/2}} = -\pi \tag{7}$$

which yields the correct result:

$$E_n \approx v_G \sqrt{2|e|B(n+1)/c} \tag{8}$$

## 2 Weyl semimetals: Fermi arcs

1. We have $\boldsymbol{d}(\boldsymbol{p}) = \pm v_F \boldsymbol{p}$; thus $\partial d_\alpha / \partial p_i = \pm v_F \delta_{\alpha j}$ and we have:

$$\Omega_i(\boldsymbol{p}) = \frac{\epsilon_{ijk}\epsilon^{\alpha\beta\gamma}}{8\pi|d|^3} d^\alpha \frac{\partial d^\beta}{\partial p_j} \frac{\partial d^\gamma}{\partial p_k} = \frac{d_i(\boldsymbol{p})}{4\pi v_F |p|^3} = \pm \frac{p_i}{4\pi|p|^3} \tag{9}$$

As the normal vector to the sphere is $\boldsymbol{n_p} = \boldsymbol{p}/|\boldsymbol{p}|$, we have $\boldsymbol{\Omega} \cdot d^2\boldsymbol{S} = \pm 1/4\pi p^2$ and thus $Q = \pm 1$. The matrix $A_{ij} = \pm v_F \delta_{ij}$, so $\det A = \pm v_F^3$ and thus $Q$ indeed coincides with the given formula.

2. The given wavefunction satisfies the boundary conditions if

$$(\hat{\sigma}_x \cos\alpha + \hat{\sigma}_y \sin\alpha)\,|\psi(z=0)\rangle = \begin{pmatrix} 0 & e^{-i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow c_1 e^{i\alpha} + c_2 e^{-i\alpha} = 0 \Rightarrow \begin{cases} c_1 = e^{-i\alpha}/\sqrt{2} \\ c_2 = e^{i\alpha}/\sqrt{2} \end{cases}$$

It solves the Schroedinger equation provided

$$\begin{pmatrix} i\kappa & pe^{-i\phi} \\ pe^{i\phi} & -i\kappa \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \Rightarrow \kappa = p, \quad ic_1 + e^{-i\phi} c_2 = 0 \Rightarrow \phi = \frac{\pi}{2} + 2\alpha \tag{10}$$

which indeed forms a ray which starts from the Dirac node.

# 3 Weyl semimetals: Landau Levels and chiral anomaly

1. The commutator reads:

$$[\hat{P}_x, \hat{P}_y] = \left[\hat{p}_x - \frac{e}{c}A_x(\boldsymbol{r}), \hat{p}_y - \frac{e}{c}A_y(\boldsymbol{r})\right] = -\frac{e}{c}\left[\hat{p}_x, A_y\right] + \frac{e}{c}\left[\hat{p}_y, A_x\right] = \frac{ie}{c}\left[\partial_x A_y - \partial_y A_x\right] = \frac{ieB}{c} \tag{11}$$

thus:

$$[\hat{\Pi}, \hat{\Pi}^\dagger] = 2i\left[\hat{P}_x, \hat{P}_y\right] = \frac{2|e|B}{c} \tag{12}$$

and therefore $\hat{a} = \sqrt{c/2|e|B} \cdot \hat{\Pi}$ form standard oscillator algebra.

2. Substitution of the given ansatz to the Schroedinger equation gives:

$$\hat{H}\,|\Psi_n\rangle = \chi v_F \begin{pmatrix} p_z & \sqrt{2|e|B/c}\hat{a} \\ \sqrt{2|e|B/c}\hat{a}^\dagger & -p_z \end{pmatrix} \begin{pmatrix} u_n\,|n-1\rangle \\ v_n\,|n\rangle \end{pmatrix} = \chi v_F \begin{pmatrix} \left(p_z u_n + \sqrt{2|e|Bn/c}v_n\right)|n-1\rangle \\ \left(\sqrt{2|e|Bn/c}u_n - p_z v_n\right)|n\rangle \end{pmatrix} \tag{13}$$

and thus we obtain following equations for $u_n$ and $v_n$:

$$\chi v_F \begin{pmatrix} p_z & \sqrt{2|e|Bn/c} \\ \sqrt{2|e|Bn/c} & -p_z \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = E_n \begin{pmatrix} u_n \\ v_n \end{pmatrix} \Rightarrow E_n^\pm = \pm\chi v_F \sqrt{p_z^2 + 2|e|Bn/c}, \quad n > 0 \tag{14}$$

For $n = 0$ we obtain:

$$E_0 = -\chi v_F p_z \tag{15}$$

It describes a mode with group velocity $v_z = -\chi v_F$, i.e. it is parallel to $z$ axis for $\chi = -1$ and anti-parallel for $\chi = +1$.

3. Introduce a magnetic length $l_B = \sqrt{c/|e|B}$. Then the density of states reads:

$$\nu(\varepsilon) = \sum_n \frac{1}{2\pi l_B^2} \int_{-\infty}^{\infty} \frac{dp_z}{2\pi} \delta\left(\varepsilon - v_F\sqrt{p_z^2 + 2/l_B^2}\right) \tag{16}$$

Only Landau Levels with $n < n^* = \varepsilon^2 l_B^2 / 2v_F^2$ contribute to this sum. Separating also explicitly the zero Landau Level, we obtain:

$$\begin{aligned} \nu(\varepsilon) &= \frac{1}{4\pi^2 l_B^2} \int_{-\infty}^{\infty} dp_z \delta\left(\varepsilon + v_F p_z\right) + \frac{1}{4\pi^2 l_B^2} \sum_{n=1}^{[n^*]} \int_{-\infty}^{\infty} dp_z \delta\left(\varepsilon - v_F\sqrt{p_z^2 + 2/l_B^2}\right) \\ &= \frac{1}{4\pi^2 l_B^2 v_F}\left(1 + 2\varepsilon \sum_{n=1}^{[n^*]} \frac{1}{\sqrt{\varepsilon^2 - 2v_F^2 n/l_B^2}}\right) = \frac{1}{4\pi^2 l_B^2 v_F}\left(1 + 2\sum_{n=1}^{[n^*(\varepsilon)]} \sqrt{\frac{n^*(\varepsilon)}{n^*(\varepsilon) - n}}\right) \end{aligned} \tag{17}$$

By comparison, in the absence of magnetic field the density of states would read (see figure)

$$\nu^{(0)}(\varepsilon) = \int \frac{d^3p}{(2\pi)^3} \delta\left(\varepsilon - v_F|p|\right) = \frac{\varepsilon^2}{2\pi^2 v_F^3} \tag{18}$$

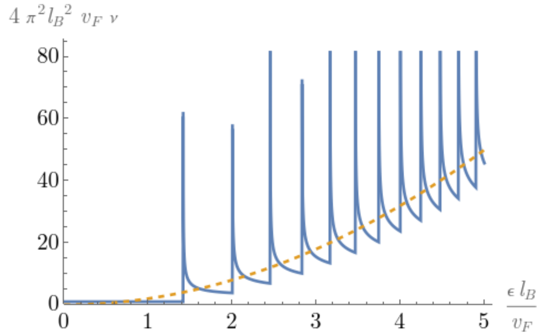


Figure 1: Blue: density of states in the presence of magnetic field, dashed orange: density of states without magnetic field

4. The total density of particles on the lowest Landau level with the Fermi-energy $\varepsilon_F = -\chi v_F p_z^*$ is given by:

$$\rho(\varepsilon_F) = \text{const} + \int_0^{\varepsilon_F} \nu(\varepsilon) d\varepsilon = \text{const} + \frac{\varepsilon_F}{4\pi^2 l_B^2 v_F} = \text{const} - \frac{\chi p_z^*}{4\pi^2 l_B^2} \tag{19}$$

where const denotes contribution from the filled Fermi see with zero Fermi energy. Then the rate of change is given by:

$$\frac{d\rho}{dt} = -\frac{\chi}{4\pi^2 l_B^2} \frac{dp_z^*}{dt} = \chi \frac{e^2}{4\pi^2 c} BE \tag{20}$$