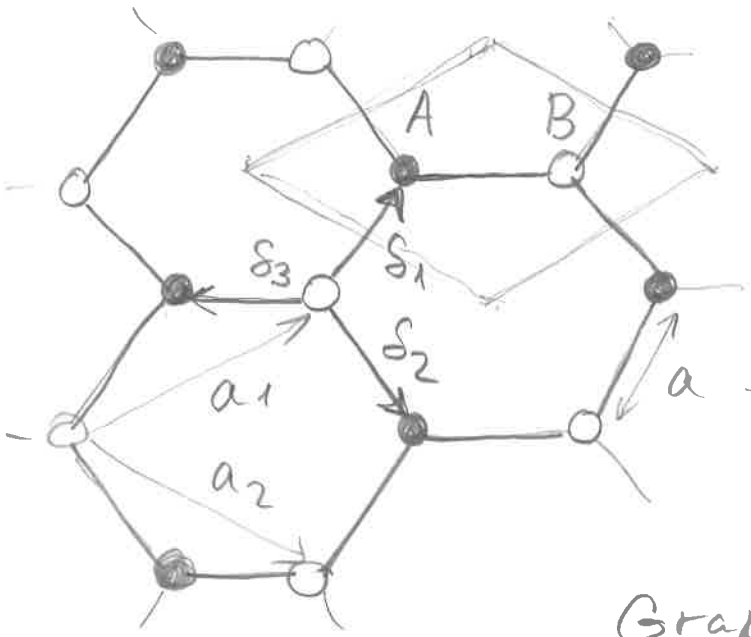
6. Graphene and 2D Dirac fermions, "gapless topology"

6.1 Hexagonal lattice



Two sublattices: A and B atoms in a unit cell

a - bond length

Graphene lattice:

Bravais lattice: triangular $\vec{a}_1, \vec{a}_2$ - primitive vectors

$$\vec{a}_1 = \frac{a}{2}\begin{pmatrix} 3 \\ \sqrt{3} \end{pmatrix} \qquad \vec{a}_2 = \frac{a}{2}\begin{pmatrix} 3 \\ -\sqrt{3} \end{pmatrix}$$

Bond vectors:

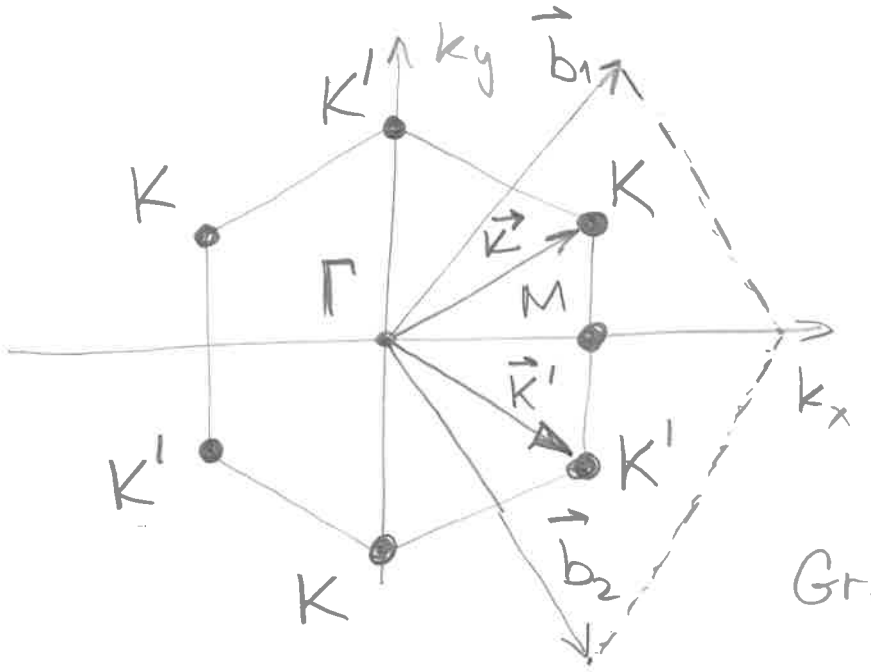
$$\vec{\delta}_1 = \frac{a}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} \qquad \vec{\delta}_2 = \frac{a}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \qquad \vec{\delta}_3 = a\begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$\vec{a}_1 = \vec{\delta}_1 - \vec{\delta}_3; \quad \vec{a}_2 = \vec{\delta}_2 - \vec{\delta}_3$

Reciprocal lattice vectors ($\vec{b}_i \cdot \vec{a}_j = 2\pi\delta_{ij}$)

$\vec{b}_i \cdot \vec{\delta}_j \neq 2\pi\delta_{ij}$

$$\vec{b}_1 = \frac{2\pi}{3a}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \qquad \vec{b}_2 = \frac{2\pi}{3a}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$



Graphene BZ

$$\vec{K} = \frac{2\pi}{3a}\begin{pmatrix} 1 \\ 1/\sqrt{3} \end{pmatrix}, \qquad \vec{K}' = \frac{2\pi}{3a}\begin{pmatrix} 1 \\ -1/\sqrt{3} \end{pmatrix}$$

$$-\vec{K} = \vec{K}' - \vec{b}_1 - \vec{b}_2 \quad \rightarrow \text{TR partners}$$

Tight-binding Hamiltonian

simplest case: nearest-neighbor hoppings t_1, t_2, t_3

$$H = \sum_{\vec{k}} \begin{pmatrix} c^\dagger_{A\vec{k}} & c^\dagger_{B\vec{k}} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & \sum_{i=1}^{3} t_i e^{i\vec{k}\vec{\delta}_i} \\ \sum_{i=1}^{3} t_i e^{-i\vec{k}\vec{\delta}_i} & 0 \end{pmatrix}}_{h(\vec{k})} \begin{pmatrix} c_{A\vec{k}} \\ c_{B\vec{k}} \end{pmatrix}$$

Bloch form

$$h_B(\vec{k}+\vec{G}) = h_B(\vec{k})$$

can be achieved by gauge transformation

$$c_{B\vec{k}} \rightarrow \tilde{c}_{B\vec{k}}\, e^{i\vec{k}\cdot\vec{\delta}_3}$$

$$\vec{G} = n_1\vec{b}_1 + n_2\vec{b}_2$$

$$H = \sum_{\vec{k}} \begin{pmatrix} \tilde{c}^{+}_{A\vec{k}} & \tilde{c}^{+}_{B\vec{k}} \end{pmatrix} h(\vec{k}) \begin{pmatrix} \tilde{c}_{A\vec{k}} \\ \tilde{c}_{B\vec{k}} \end{pmatrix}$$

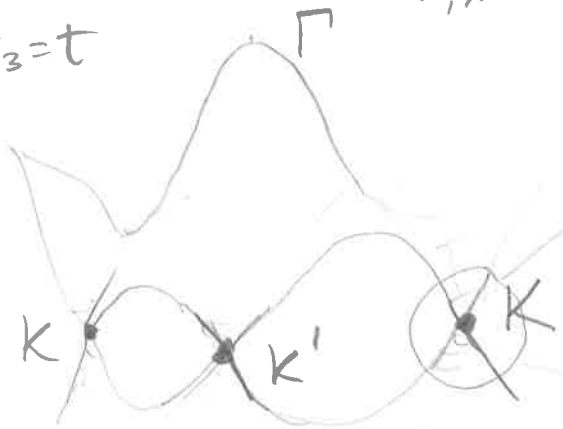
with

$$h_B(\vec{k}) = \begin{pmatrix} 0 & -t_1 e^{i\vec{k}\vec{a}_1} - t_2 e^{i\vec{k}\vec{a}_2} - t_3 \\ -t_1 e^{-i\vec{k}\vec{a}_1} - t_2 e^{-i\vec{k}\vec{a}_2} - t_3 & 0 \end{pmatrix}$$

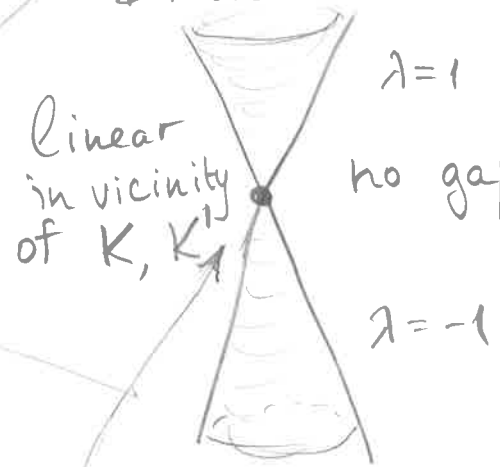
Chiral symmetry!

Dispersion: $\epsilon_{\vec{k},\lambda} = \lambda t \sqrt{3 + 2\sum_{i=1}^{3} \cos(\vec{k}\vec{a}_i)}$, $\lambda = \pm 1$

$t_1 = t_2 = t_3 = t$



Dirac cone



Eigenfunctions: $\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \lambda \cdot e^{-i\xi_k} \end{pmatrix}$, Dirac node

where ξ_k is the phase of h_{12}.

Again isotropic hopping $t_1 = t_2 = t_3 = t$:

$$h(\vec{K}+\vec{q}) = (q_x \sigma_x + q_y \sigma_y) v_G$$

$\vec{\sigma}$ - vector of Pauli matrices in sublattice space

$\vec{k} = \vec{K} + \vec{q}$ near K-point

$|q| \ll |\vec{K}|$

$v_G = \frac{3ta}{2\hbar}$; in what follows, set $\hbar = 0$, $a = 1$

with $a = 0.14\,\text{nm}$ for graphene.

$v_G \approx 10^6\,\text{m/s}$

Near K'-point:

$$h(-\vec{K}+q) = (-q_x \sigma_x + q_y \sigma_y) v_G$$

Two Dirac fermions (at K and K')

with $\epsilon_{\vec{q},\lambda} = \lambda \cdot v_G |\vec{q}|$

no gap, linear dispersion, independent of "valley" (K, K')

Positions of Dirac nodes:

$$3 + 2\sum_{i=1}^{3} \cos(\vec{k}\cdot\vec{a}_i) = 0, \quad \vec{k} = \vec{K}, \vec{K}_1$$

Simplest Hamiltonian
(only nearest-neighbor hopping with C_3 symmetry, same on-site energies...)

$\Rightarrow$ semimetal

What perturbations open a gap?
Semimetal $\xrightarrow[\text{perturbation}]{}$ insulator

6.2 Symmetries of graphene, stability of Dirac node

A-B structure with off-diagonal Hamiltonian $\Rightarrow$ chiral symmetry (sublattice symmetry)

$U_S = \sigma_z$, $\quad U_S^\dagger h(\vec{k}) U_S = -h(\vec{k})$

Time reversal symmetry:
always present for real hoppings
(independent of their spatial structure)

T: $\left[h(\vec{k})\right]^* = h(-\vec{k})$

Inversion symmetry (parity)

I: $(x,y) \to (-x,-y)$

Previously denoted by P

$$I\, c_{\vec{k}\alpha}\, I^{-1} = \sigma_{x,\alpha\alpha'} c_{-\vec{k},\alpha'}$$

$$IHI^{-1} = H \iff h(\vec{k}) = \sigma_x h(-\vec{k}) \sigma_x$$

Dirac Hamiltonian:

TRS: $\left[h(\vec{K}+\vec{q})\right]^* = h(-\vec{K}-\vec{q}) = v_G (q_x\sigma_x + q_y\sigma_y)^*$

$$h_K(\vec{q}) \equiv v_G(q_x\sigma_x + q_y\sigma_y)$$

$$h(-\vec{K}+\vec{q}) = v_G(-q_x\sigma_x + q_y\sigma_y) \equiv h_{K'}(\vec{q})$$

$$h_K(\vec{q}) = h^*_{K'}(-\vec{q})$$

TRS symmetry relates K and K' points

By itself, TRS symmetry does not protect Dirac fermions from opening a gap:

$$h_K(q) = v(q_x\sigma_x + q_y\sigma_y + m\sigma_z)$$

$$h_{K'}(q) = v(-q_x\sigma_x + q_y\sigma_y + m\sigma_z)$$

→ TRS preserved, but gap opens

This type of mass corresponds to different on-site energies of A and B atoms

→ momentum-independent term $m\sigma_z$

(example: boron nitride BN)
→ gapped insulator
breaks inversion symmetry

If masses in two valleys have opposite signs,

$$h_K(q) = v(q_x\sigma_x + q_y\sigma_y + m\sigma_z)$$

$$h_{K'}(q) = v(-q_x\sigma_x + q_y\sigma_y - m\sigma_z)$$

inversion symmetry is preserved

Hence, inversion symmetry by itself also does not protect dispersion from gap opening.

Taken separately, TRS and IS link $\vec{k}$ and $-\vec{k}$

When both TRS and IS are present, they impose constrains at each $\vec{k}$ separately:

$$h^*(-\vec{k}) \overset{\text{TRS}}{=} h(\vec{k}) \overset{\text{IS}}{=} \sigma_x h(-\vec{k}) \sigma_x$$

$$\Rightarrow h(\vec{k}) = \sigma_x h^*(\vec{k}) \sigma_x$$

$$(T \cdot I)\, h(\vec{k}) (T\, I)^{-1} = h(\vec{k})$$

General 2×2 Hamiltonian

$$H = \vec{d}(\vec{k})\cdot\vec{\sigma} + \epsilon(\vec{k})\cdot\mathbb{1}$$

T+I:

$$\vec{d}\cdot\vec{\sigma} + \epsilon = \sigma_x(\vec{d}\cdot\vec{\sigma} + \epsilon)\sigma_x$$

$$= d_x\sigma_x + d_y\sigma_y - d_z\sigma_z + \epsilon$$

$$\Rightarrow d_z(k) = -d_z(k) = 0$$

no σ_z term can arise if both TRS and inversion are preserved

Any σ_x and σ_y perturbations (if small) only shift Dirac points

$$\tilde{h} = v(k_x\sigma_x + k_y\sigma_y) + a_x\sigma_x + a_y\sigma_y$$

$\rightarrow$ nodes at $k_x = -\frac{a_x}{v}$, $k_y = -\frac{a_y}{v}$

no gap

Inversion plus time reversal symmetries make a single Dirac node **locally stable**

(large perturbation may open the gap)

Dirac fermions carry a vortex in the Bloch wavefunction: their Berry phase equals $\pm\pi = \pi \det \hat{V}$, where $\hat{V}$ is defined through $h = q_i V_{ij} \sigma_j$

Lack of σ_z is important, codimension of crossing is 2.

A is a square matrix.

Berry phases are opposite in K and K'

Global stability:

if the Hamiltonian has C_3 symmetry (on top of T and I), no perturbation can open the gap

Breaking of C_3 symmetry:

Dirac points move away from K, K'

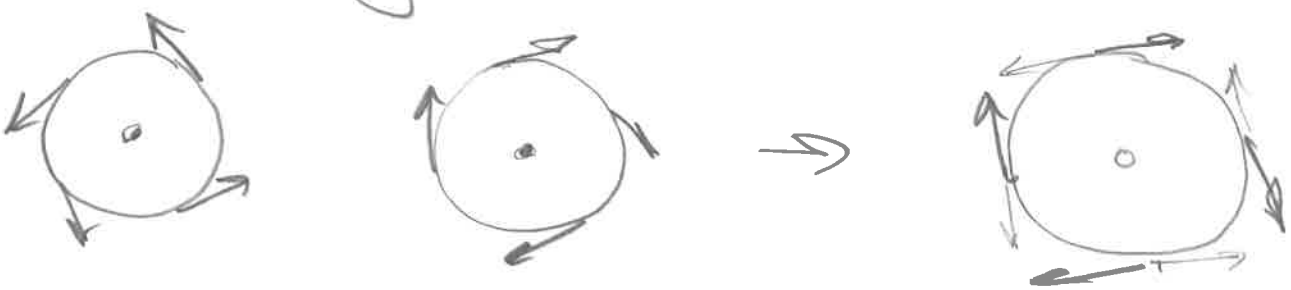
High anisotropy → gap

for example: $t_3 \gg t_1, t_2$

$$h(k) \simeq -t_3 \sigma_x, \text{ gap } 2|t_3|$$

Mechanism:

merging two Dirac nodes



$-K_0$ K_0

Touching nodes: dispersion is linear in one direction but quadratic in the other

6.3 Edge states in graphene

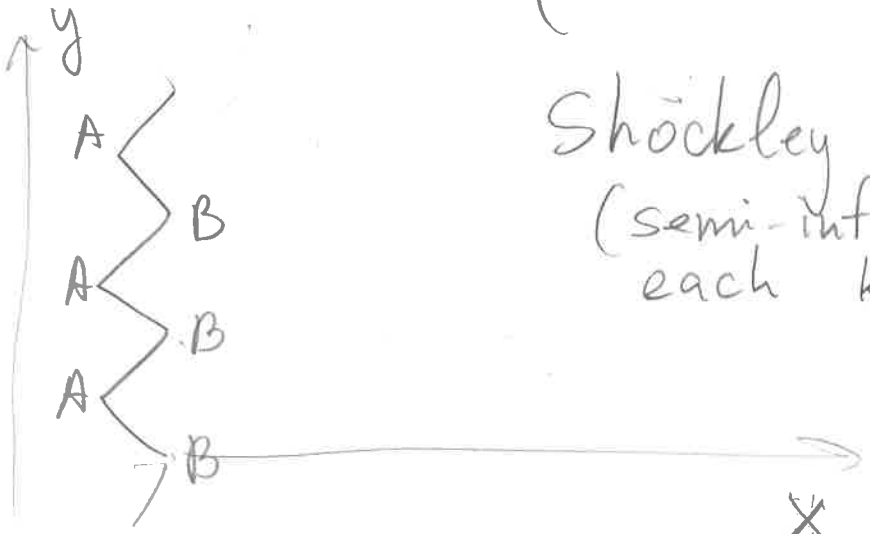
$$|\Psi\rangle = \sum_R \left(A|\vec{R}\rangle + B|\vec{R}-\vec{\delta}_3\rangle \right) e^{i\vec{k}\vec{R}}$$

$$E\langle\vec{R}|\Psi\rangle = \langle\vec{R}|\hat{H}|\Psi\rangle$$

$t_1 = t_2 = t_3 = t = 1$: $\langle\vec{R}|\hat{H}|R\pm\delta_j\rangle = 1$

$a = 1$

$$EA = B\left(1 + e^{-i\sqrt{3}k_y} + e^{-i\frac{\sqrt{3}}{2}k_y} e^{-i\frac{3}{2}k_x}\right)$$



Shöckley model
(semi-infinite SSH for each k_y)

$$\tilde{t} = 1 + e^{-i\sqrt{3}k_y} \qquad \tilde{t}' = e^{-i\frac{\sqrt{3}}{2}k_y}$$

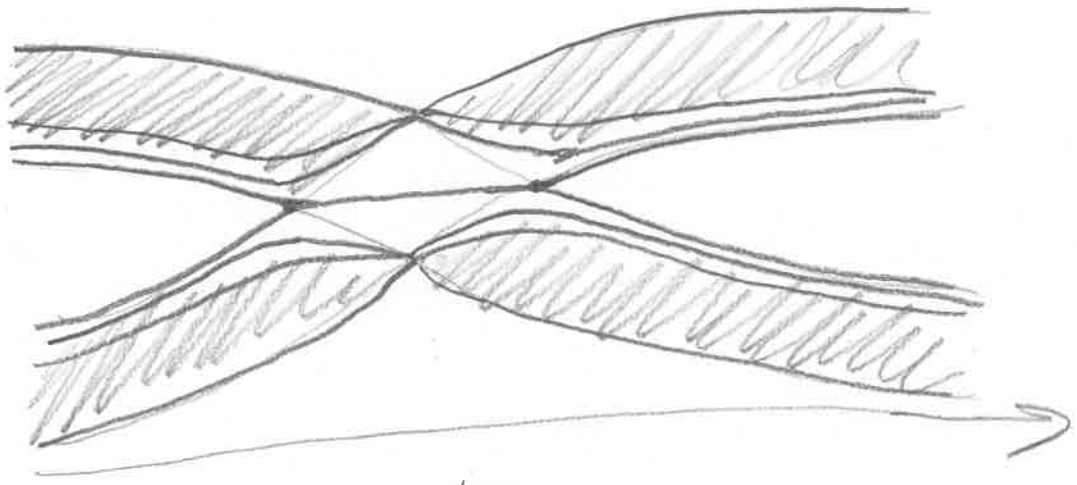
$$k = \frac{3}{2}k_x$$

Zero-E edge state when

$$|t| < |t'| \iff |1 + e^{-i\sqrt{3}k_y}| < 1$$

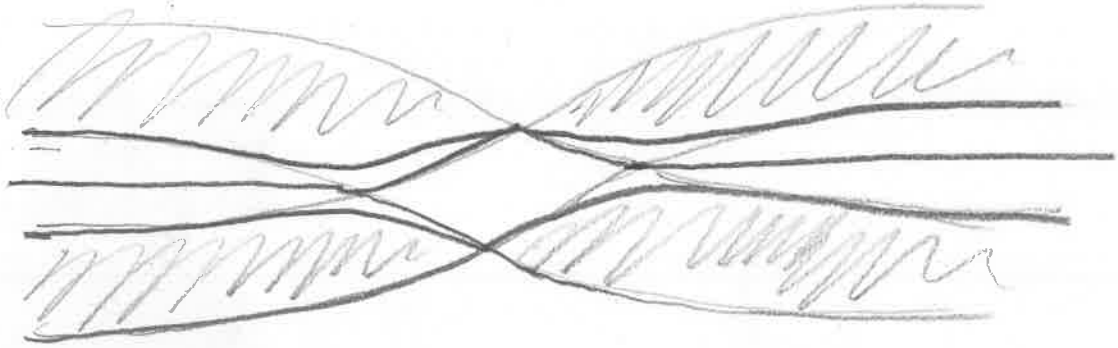
$$\iff |k_y| > \frac{2\pi}{3\sqrt{3}}$$

Even number of sites,
boundaries A and B

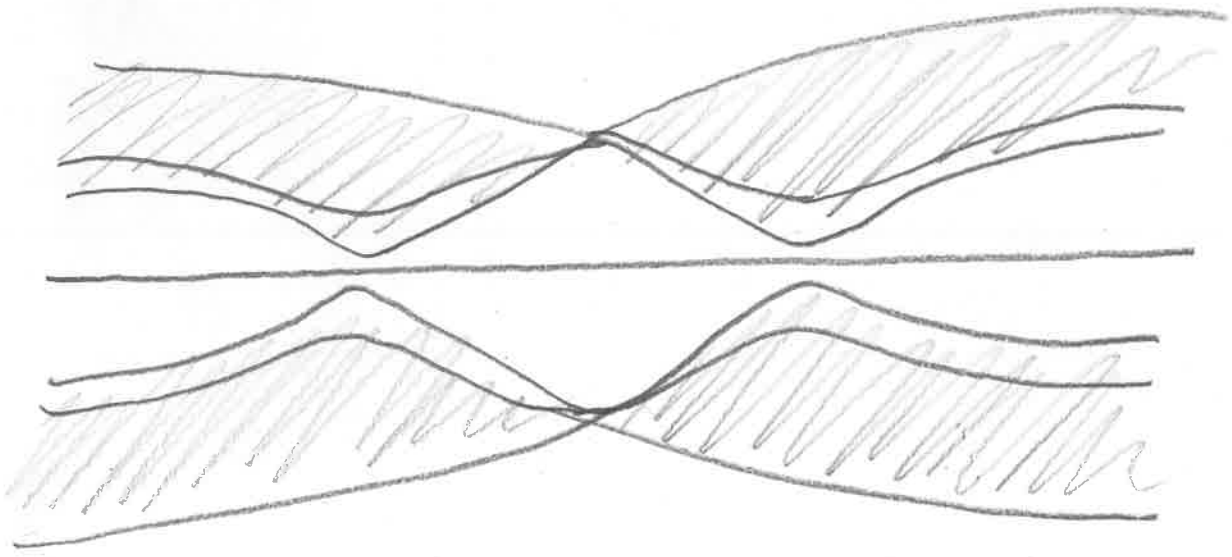


k_y

B and A



Odd number of sites A and A



Existence of edge states

Consider $\hat{H}(\vec{k}) = \vec{h}(\vec{k}) \cdot \vec{\Gamma}$

$\vec{\Gamma} = (\ldots, \Gamma_i, \ldots)$ set of Hermitian matrices

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij}\, \mathbb{I}$$

↖ identity matrix

$\vec{h} = (\ldots, h_i, \ldots)$

Trace of H is zero.

Indeed, $\Gamma_j^2 = \mathbb{I}$

$\Gamma_j \Gamma_i \Gamma_j = -\Gamma_i$ for $\Gamma_i \neq \Gamma_j$

$\Rightarrow \mathrm{Tr}\{\Gamma_j \Gamma_i \Gamma_j\} = \mathrm{Tr}\{\Gamma_j \Gamma_j \Gamma_i\}$

$= \mathrm{Tr}\{\Gamma_i\} = -\mathrm{Tr}\{\Gamma_i\}$

$\Rightarrow \mathrm{Tr}\{\Gamma_i\} = 0 \Rightarrow \mathrm{Tr}\, H = 0$

$\hat{H}^2(\vec{k}) = \vec{h}^2(\vec{k})\, \mathbb{I}$

$\Rightarrow E(\vec{k}) = \pm |\vec{h}(\vec{k})|$

Consider a semi-infinite system,
let $k_\perp$ be the normal component of the momentum $\vec{k} = (\vec{k}_\parallel, k_\perp)$.

$$\vec{h}(\vec{k}) = \vec{b}_0 + 2\vec{b}_c \cos k_\perp + 2\vec{b}_s \sin k_\perp$$

$\vec{b}_0, \vec{b}_c, \vec{b}_s$ are functions of $\vec{k}_\parallel$

Keeping $\vec{k}_\parallel$ fixed, $\vec{h}(k_\perp)$ traces an ellipse centered at $\vec{b}_0$ when changing $k_\perp$ from $-\pi$ to π

Ellipse's plane is defined by vectors $\vec{b}_c$ and $\vec{b}_s$

Split $\vec{b}_0 = \vec{b}_\parallel + \vec{b}_\perp$ w.r.t. ellipse's plane

← linear combination of $\vec{b}_c$ and $\vec{b}_s$

Existence condition:

Edge states exist if the projection of the origin onto the ellipse plane is contained inside the ellipse

$\Leftrightarrow$ $\vec{b}_\parallel + 2\vec{b}_c \cos k_\perp + 2\vec{b}_s \sin k_\perp$ contains the origin

Energy of the edge state

$$E_{es}(\vec{k}_\parallel) = \pm |\vec{b}_\perp|$$

6.4 Landau levels in graphene, quantum Hall effect

Graphene in magnetic field:

$$H = v_G \vec{\sigma} \cdot (\vec{p} - e\vec{A})$$

set $\hbar, c = 1$

Landau gauge $\vec{A} = -yB\vec{e}_x$; $l_0^2 = \frac{1}{|e|B}$

$$H = v_G \begin{pmatrix} 0 & p_x - ip_y - \frac{y}{l_0^2} \\ p_x + ip_y - \frac{y}{l_0^2} & 0 \end{pmatrix}$$

$$= v_G \begin{pmatrix} 0 & \Pi \\ \Pi^+ & 0 \end{pmatrix},$$

$$[\Pi, \Pi^+] = \frac{2}{l_0^2} = -ie(\partial_x A_y - \partial_y A_x)$$

without specifying the gauge

Ladder operators:

$$a = \frac{l_0}{\sqrt{2}}\Pi, \quad a^+ = \frac{l_0}{\sqrt{2}}\Pi^+, \qquad [a, a^+] = 1$$

$$H = \frac{\sqrt{2}v_G}{l_0}\begin{pmatrix} 0 & a \\ a^+ & 0 \end{pmatrix} = \omega_c \begin{pmatrix} 0 & a \\ a^+ & 0 \end{pmatrix}$$

ω_c: cyclotron Frequency

$$H^2 = \omega_c^2 \begin{pmatrix} a^+a + 1 & 0 \\ 0 & a^+a \end{pmatrix}$$

Eigenstates:

$$|\Psi_{\lambda,n}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |\Phi_{n-1}\rangle \\ \lambda |\Phi_n\rangle \end{pmatrix}, \qquad n>0, \quad \lambda = \pm 1$$

$|\Phi_n\rangle$ – harmonic oscillator states

The eigenfunctions are the same as in parabolic band

$$X_n(x) = e^{-\frac{x^2}{2\ell_0^2}} H_n\left(\frac{x}{\ell_0}\right),$$ see 4.1 (p. 40 of lectures 1–6)

(here we disregard degeneracy, interested only in energy levels; degeneracy can be restored following the calculation in sec. 4.1)

Eigenenergies:

$$E_{\lambda,n} = \lambda \omega_c \sqrt{n}, \qquad n \in \mathbb{N}_{>0}$$

$n=0$

$$|\Psi_0\rangle = \begin{pmatrix} 0 \\ |\Phi_0\rangle \end{pmatrix}$$

← lives in one sublattice

$$E_0 = 0$$

(in a given Dirac cone; in the other sublattice in the other cone)

When the Dirac mass is added

$$H_K = \begin{pmatrix} m & \omega_c a^+ \\ \omega_c a & -m \end{pmatrix}, \qquad H_{K'} = \begin{pmatrix} m & \omega_c a \\ \omega_c a^+ & -m \end{pmatrix}$$

$$E_{\lambda,n} = \lambda \sqrt{\omega_c^2 n + m^2}, \qquad n = 1, 2 \ldots, \quad \lambda = \pm 1$$

for $n=0$, $\lambda = 1$ at K and $\lambda = -1$ at K'

Landau levels are no longer equidistant, getting "closer together" at higher energies

(density of states grows $\rho(E) \propto E$,
integrated density of states $\propto E^2$;
for fixed number of particles,
the number of Landau levels $n \propto E^2$
$\Rightarrow E_n \propto \sqrt{n}$]

Semiclassical derivation – Exercise

($\sqrt{n}$ instead of $\sqrt{n+1/2}$
– effect of Berry phase)

Hall conductivity

insulators:

Section 4.3
P. 45
of lectures 1-6

$$\varkappa_i = \frac{\Phi_i}{\hbar} \frac{e}{c L_i}$$

$$\sigma_{yx} = \frac{e^2}{\hbar A} \Omega^{(0)}_{\varkappa_x \varkappa_y}$$

← Berry curvature of $|\Psi_0\rangle$

$$= \frac{e^2}{h} \cdot C$$

← Chern number

$$C = \sum_{\text{occupied } i} C_i$$

← sum over occupied bands
(Fermi level in the gap)

TKNN

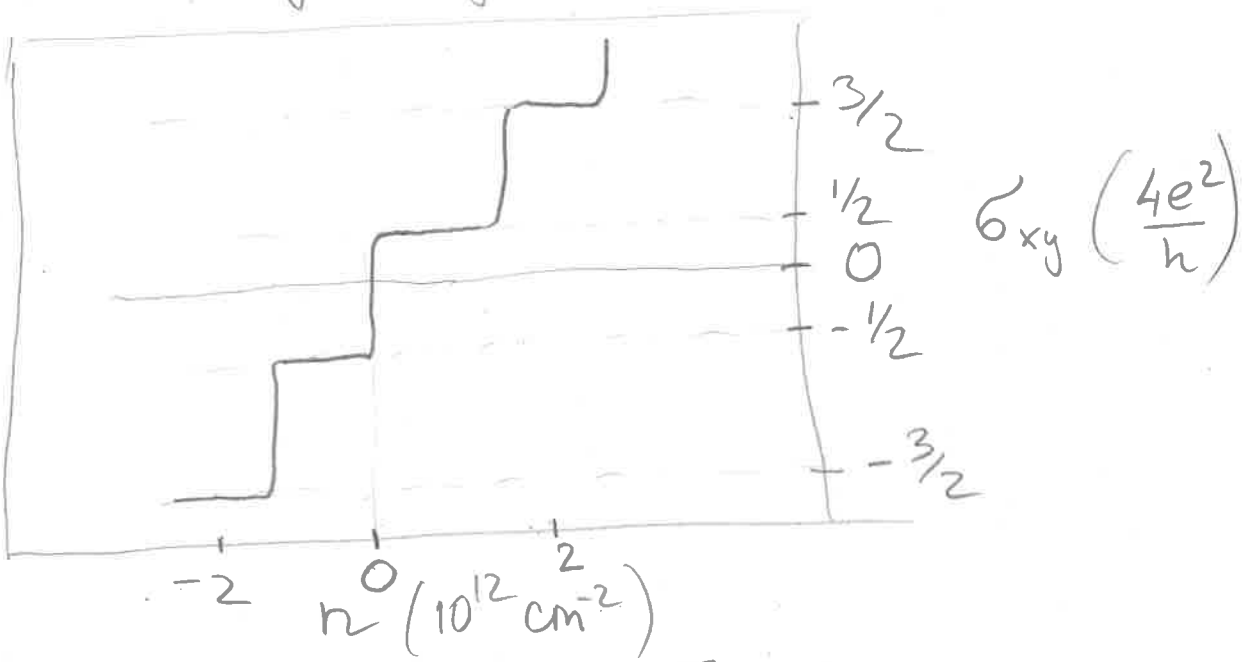
Each electron Landau level contributes 1 in units of e^2/h to σ_{xy}, each hole LL -1

$n=0$ Landau level is guaranteed by particle-hole symmetry it is half-filled $\Rightarrow$ for one Dirac node

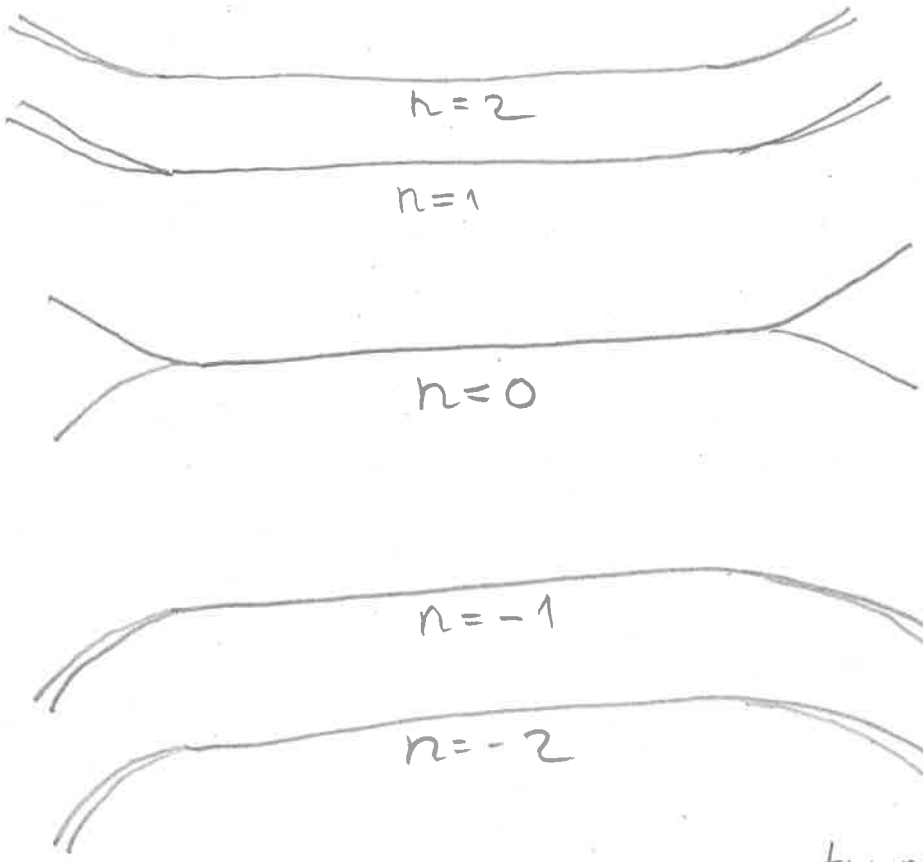
$$\sigma_{xy} = \frac{1}{2} e^2/h$$ (half-integer quantization)

However, the second Dirac point contributes equally

$$\sigma_{xy} = \underset{\text{spin degeneracy}}{2} \cdot \underset{K+K' \text{ points}}{2} \cdot \left(n + \frac{1}{2}\right) \frac{e^2}{h} = \left(\pm 2, \pm 6, \pm 10 \ldots\right) \frac{e^2}{h}$$



$B = 14\,T$



two branches of edge modes at each edge (one from K, one from K')

6.5 Weyl semimetals ("3D graphene")

$$H = \pm v_F \vec{k}\cdot\vec{\sigma} \qquad \vec{k} = (k_x, k_y, k_z)$$

Pairs of Weyl points (nodes) — sources and drains of Berry curvature

Nodes are robust with respect to opening the gap:

adding $\vec{a}\cdot\vec{\sigma}$ shifts the nodes

Topological protection: 2π Berry flux of each node.

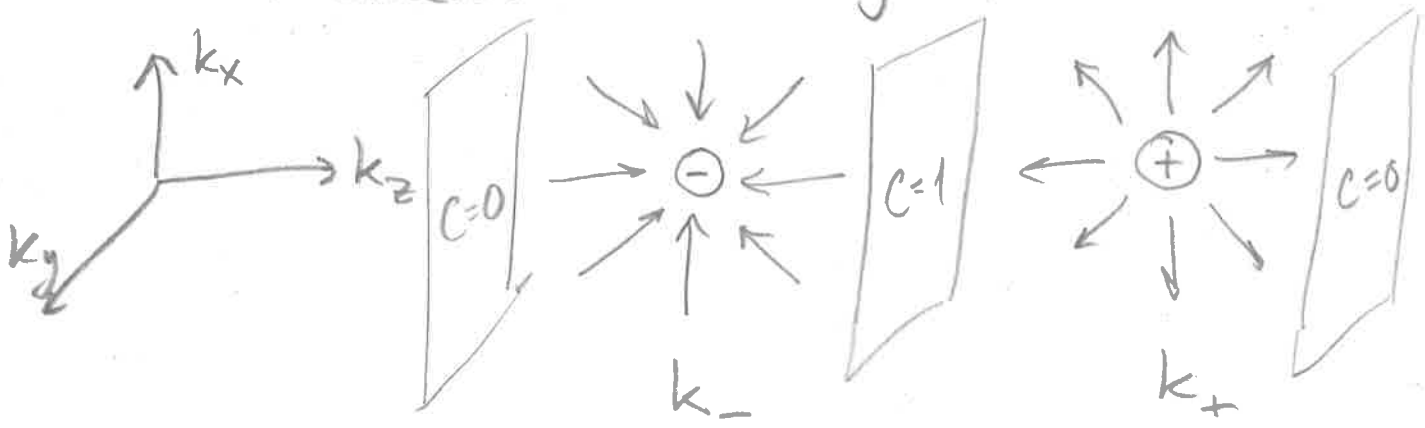
Surface (edge) states: Fermi arcs (exercise)

Fermi arcs

Assume a single pair of Weyl points with opposite chiralities

(TRS is broken; if SIS is broken but TRS is intact there are two such pairs: TRS transformation $\vec{K} \leftrightarrow -\vec{K}$ with the same chirality; the net chirality must be zero, as total Berry flux through $BZ = 0$
$\Rightarrow$ number of Weyl points is $4n$, $n \in \mathbb{N}$ if TRS is preserved)

Choose z-direction along the difference of momenta of Weyl points: $k_z = k_\pm$



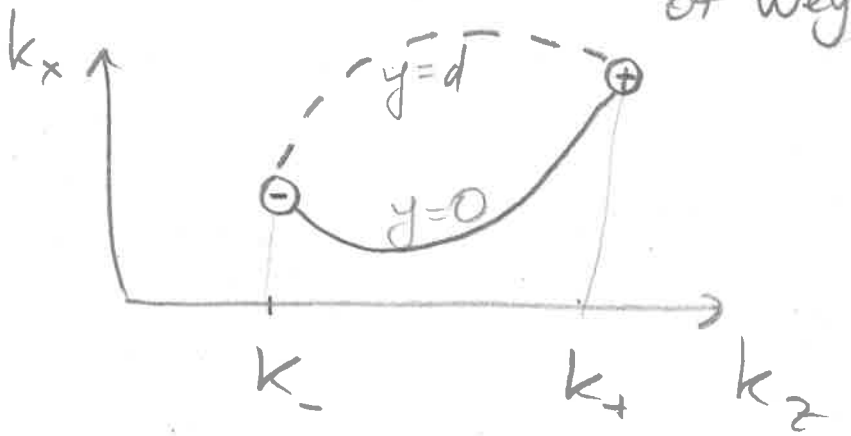
Treat k_z as a parameter: for each slice of k_z there is a 2D band structure that depends on (k_x, k_y).
Except for $k_z = k_\pm$, there is a gap in 2D bands at Fermi energy.

Define a k_z-dependent Chern number $C(k_z)$; it can only change at $k_z = k_\pm$
(effectively the gap of 2D band is $m = v_F k_z$)

$C(k_z)$ must differ in $k_- < k_z < k_+$
and in the complement of this range
$\Rightarrow$ somewhere $C(k_z) \neq 0$

Consider $C=1$ for $k_- < k_z < k_+$
and $C=0$ otherwise
$\Rightarrow$ gapless surface state for $k_- < k_z < k_+$.

Fermi arcs connect projections of Weyl points to $k_x k_z$ plane



infinite crystal in x and z directions, two boundaries separated in y-direction by d

When approaching the nodes, surface states penetrate deeper and deeper to the bulk; at the circles, they are completely mixed with with bulk states: the Fermi arcs on opposite boundaries are connected at Weyl points through the bulk.

Weyl semimetals in magnetic or electric field: chiral anomaly
$N_\pm$ are not conserved separately
(exercise)

6.6. Relation to Berry phase (σ_{xy} in 2D Dirac cone)

Return to (massive) Dirac fermions in two dimensions

$$h(\vec{k}) = \vec{d}(\vec{k})\cdot\vec{\sigma},$$

$d_z = 0$ - gapless Hamiltonian

$$E_\pm = \pm\sqrt{d_x^2 + d_y^2 + d_z^2} = \pm|\vec{d}| \equiv d$$

$$\Psi_+ = \frac{1}{\sqrt{2d(d+d_z)}}\begin{pmatrix} d_z + d \\ d_x - i d_y \end{pmatrix}$$

$$\Psi_- = \frac{1}{\sqrt{2d(d-d_z)}}\begin{pmatrix} d_z - d \\ d_x - i d_y \end{pmatrix}$$

normalized orthogonal eigenstates

Berry connection for the lower band

$$A_i^{(-)}(\vec{k}) = i\langle \Psi_- | \partial_{k_i} | \Psi_- \rangle$$

$$= -\frac{1}{2d(d-d_z)}\left(d_y \partial_i d_x - d_x \partial_i d_y\right)$$

(Lecture 1-6: the same was calculated for a single spin in polar coordinates)

Berry curvature: $\Omega_{ij}^{(-)} = \frac{1}{2d^3}\epsilon_{abc} d_a \partial_i d_b \partial_j d_c$

massive Dirac fermion : ($v_G = 1$)

$$d_z = m, \quad d_x = k_x, \quad d_y = k_y$$

$$A_x^{(+)} = \frac{-k_y}{2\sqrt{k^2+m^2}\left(\sqrt{k^2+m^2}+m\right)}$$

$$A_y^{(+)} = \frac{k_x}{2\sqrt{k^2+m^2}\left(\sqrt{k^2+m^2}+m\right)}$$

Consider Fermi energy in the upper band, $\epsilon_F > m$

$\Rightarrow$ Fermi surface

$$\int_{FS} d\vec{k}\cdot\vec{A}^{(+)} = \int_0^{2\pi} k_F\, d\theta\, A_\theta^{(+)}$$

$$= \int_0^{2\pi} k_F\, d\theta \left(A_y^{(+)} \frac{k_x}{k_F} - A_x^{(+)} \frac{k_y}{k_F} \right)$$

$$= \int_0^{2\pi} \frac{k_F^2}{2\sqrt{k_F^2+m^2}\left(\sqrt{k_F^2+m^2}+m\right)}\, d\theta = \pi \frac{k_F^2}{\sqrt{k_F^2+m^2}\left(\sqrt{k_F^2+m^2}+m\right)}$$

$$\underset{m \ll k_F}{\approx} \pi - \pi \frac{m}{k_F}$$

$m = 0$: Berry phase $= \pi$ (TRS invariant 0 or π)

General Dirac Hamiltonian

$$h(\vec{k}) = k_i V_{ij} \sigma_j$$

V_{ij} a 2×2 matrix

$$d = \sqrt{(V_{1i} k_i)^2 + (V_{2i} k_i)^2}, \quad E_\pm = \pm d$$

Berry potential

$$\mathcal{A}_i(\vec{k}) = -\frac{1}{2d^2}\left(V_{2j} k_j V_{1i} - V_{1j} k_j V_{2i}\right)$$

Berry phase:

$$\int_0^{2\pi} d\varphi \left(\mathcal{A}_y k_x - \mathcal{A}_x k_y\right) = \int_0^{2\pi} d\varphi \frac{k^2}{2d^2} \mathrm{Det}\,\hat{V}$$

$$\int_0^{2\pi} d\varphi \frac{1}{a\cos^2\varphi + b\sin^2\varphi + C\cos\varphi\sin\varphi} = \frac{4\pi}{\sqrt{4ab - c^2}}$$

for $c^2 < 4ab$

$$\Rightarrow \int_{FS} d\vec{k}\,\vec{\mathcal{A}} = \pi \cdot \mathrm{sign}(\mathrm{Det}\,\hat{V})$$

Massive Dirac Hamiltonian

Berry curvature

$$\Omega_{xy} = \frac{1}{2(m^2+k^2)^{3/2}}\, m$$

ϵ_F in the gap

$$\sigma_{xy} = \frac{e^2}{h}\,\frac{1}{2\pi}\int d^2k\,\Omega_{xy}$$

$$= \frac{e^2}{h}\,\frac{m}{4}\int_0^\infty dx\,\frac{1}{(m^2+x)^{3/2}}$$

$$= \frac{e^2}{h}\,\frac{\mathrm{sign}(m)}{2}$$

Chern number $C = \pm\frac{1}{2}$
for a single Dirac point

Dirac points come in pairs
in any lattice model
(Nielsen-Ninomiya theorem)

The contribution of "spectator fermions"
away from the nodal points
(high-energy fermions) produces another $1/2$

Gap closing and reopening (sign change
of m) $\rightarrow$ jump in σ_{xy} by $1\cdot\frac{e^2}{h}$.