9. Anderson localization and topology

In this Chapter, we will discuss the interplay of Anderson localization in disordered systems and topology in symmetry-protected topological phases. In particular, we will show how analyzing the topological protection from Anderson localization one can derive the periodic table of topological insulators.

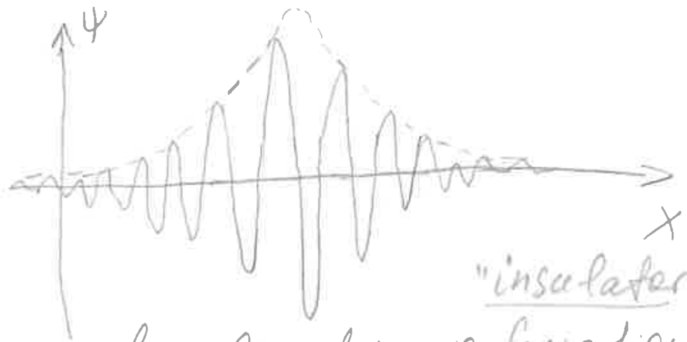
9.1. Anderson localization

Schrödinger equation in a <u>random</u> potential $U(\vec{r})$

$$\left[-\hbar^2\frac{\vec{\nabla}^2}{2m}+U(\vec{r})\right]\psi=E\psi$$

delocalized wave function

"<u>metal</u>"



"<u>insulator</u>"

localized wave function

$$|\psi|^2\sim\exp\left\{-|\vec{r}-\vec{r}_0|/\xi\right\}$$

ξ – localization length

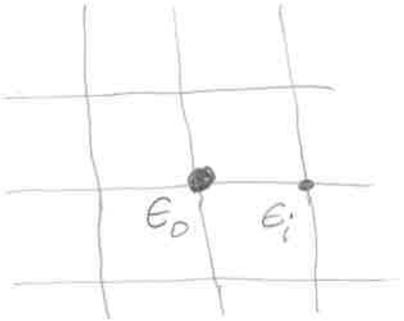
Tight-Binding model: quantum particle moving on a lattice: connectivity K, nearest-neighbor hopping V, disorder W:

$$H=\sum_i \epsilon_i c_i^\dagger c_i+\sum_{\langle ij\rangle}V\left(c_i^\dagger c_j+c_j^\dagger c_i\right)$$

ϵ_i – random, distribution width W

Anderson 1958: (Nobel Prize 1977)

Proof of localization for $W>W_c\sim V\cdot K\ln K$ by proving convergence of perturbative expansion around the localized limit



$|\epsilon_i - \epsilon_0| \lesssim W$

W/K – typical spacing of random energies ϵ_i of sites directly coupled to a given site 0.

$V \ll W/K \rightarrow$ hybridization suppressed
$\rightarrow$ Anderson localization

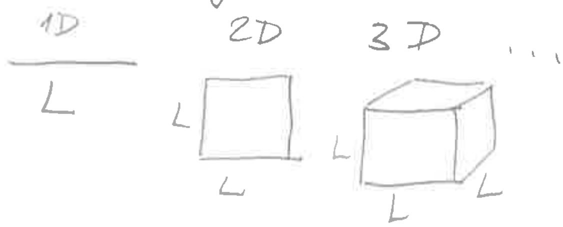
Scaling theory of localization

(Abrahams, Anderson, Licciardello, Ramakrishnan 1979)

Based on earlier ideas on connection between Anderson localization and scaling theory of phase transitions.

Scaling variable: dimensionless conductance $g = \frac{G}{e^2/h}$

G – conductance of a d-dimensional system with all linear dimensions $= L$



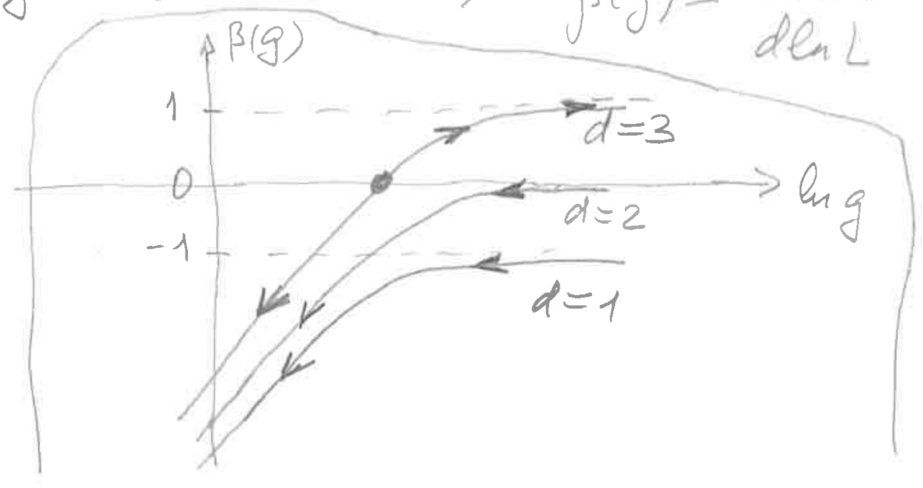
* Classically ("metallic") $G = \sigma L^{d-2}$

σ – conductivity

$\rightarrow \beta(g) \equiv \dfrac{d \ln g}{d \ln L} = d-2$

* Anderson-localized ("insulating" phase/regime)

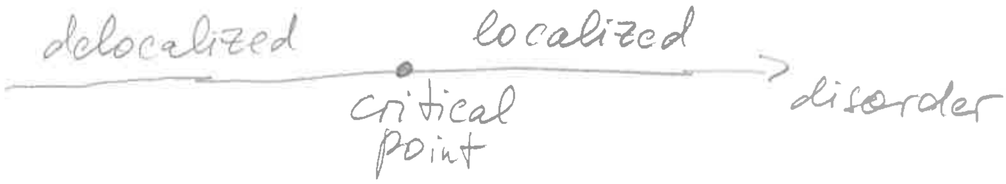
$g \sim e^{-L/\xi} \quad \rightarrow \quad \beta(g) \equiv \dfrac{d \ln g}{d \ln L} = \ln g$



connect two limits in the simplest way

$d \leq 2$ — Anderson localization for any disorder (in the limit $L \to \infty$)

$d > 2$ Anderson metal–insulator transition



Justification of the scaling theory of localization: field-theory approach (non-linear σ-model), renormalization group — see below.

9.2. Quantum Hall effect and Anderson localization

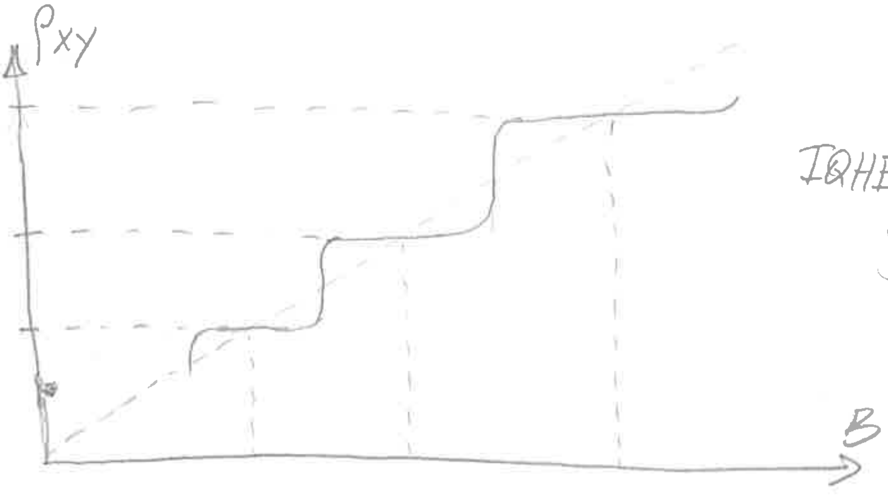
See discussion of IQHE in Sec. 42, where there was a figure with Landau-level energies (LL)

Key parameter that can be changed in experiment $\nu = \frac{hc}{e}\frac{n_e}{B}$ — filling factor of LL

Changing ν, one observes plateaus with quantized

$\sigma_{xy} = \rho_{xy}^{-1} = n\frac{e^2}{h}$, $n \in \mathbb{Z}$ $\qquad$ $\mathbb{Z}$ topological insulator

$\sigma_{xx}, \rho_{xx} = 0$

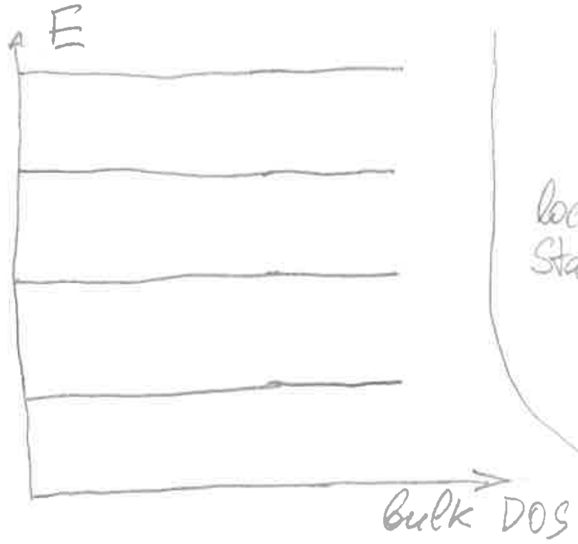


IQHE Plateaus

$$\rho_{xy} = \frac{1}{n}\frac{h}{e^2} \quad n \in \mathbb{Z}$$

middle of plateaus: $\nu = n$

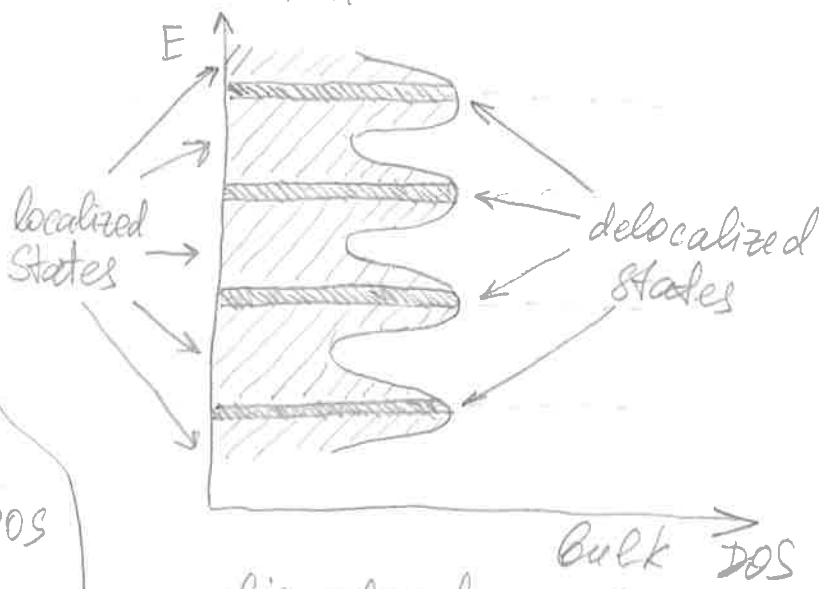
In a clean system, the chemical potential (E_F) would jump from one LL to the next one at $\nu = n$ $\rightarrow$ there would be no plateaus as a function of $\nu \propto n e/B$. The plateaus are observed due to localization



clean system

$$DOS = \frac{eB}{hc}\sum_n \delta(E - E_n)$$

E_n - Landau level energies



disordered system: regions of localized states $\rightarrow$ IQHE plateaus

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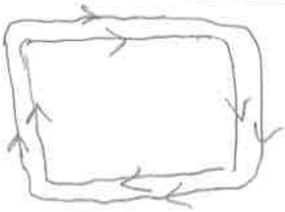
In the presence of disorder, nearly all states are localized. There are only narrow (in energy) regions of delocalized states (at the middle of LL). The width of these regions go $\rightarrow 0$ in the thermodynamic limit (in the presence of interaction: at temperature $T \rightarrow 0$).

IQHE is a paradigmatic example of topology preventing from Anderson localization:

* edge states of the topological phases.



→ 1D system.

Normally, 1D system is Anderson-localized by disorder. But here localization is impossible: chiral system, no backscattering. This is a manifestation of topology

* bulk (2D). Normally, all states in a 2D system (of this symmetry class) are Anderson-localized. But here we should have some delocalized states in the bulk, as is obvious from Laughlin's argument (sec. 4.2): Otherwise, a transfer of charge from one edge to the other would be impossible. Indeed, states at some particular energies (in the middle of LL for not too strong disorder) remain delocalized due to topology.

Inspecting the mechanism of this topological protection from localization in the field-theory approach to Anderson localization and extending it to other symmetry classes and spatial dimensionalities will allow us to rederive the periodic table, as discussed below.

9.3. Field theory of Anderson localization

As a motivation, consider the Green function (retarded) $G^R(\vec r, t; \vec r' t')$. It has a meaning of the amplitude of probability to find a particle at point $\vec r$ at time t if it was at $\vec r'$ at time t', with $t > t'$.

The corresponding probability is (R – retarded, A – advanced)

$$|G^R(\vec r t, \vec r' t')|^2 = G^R(\vec r t; \vec r' t') G^A(\vec r' t'; \vec r t) \equiv \Pi(\vec r t; \vec r' t')$$

Time-independent Hamiltonian $\hat H$ →

$$\to G^R(\vec r t; \vec r' t') = G^R(\vec r, \vec r'; t - t') \text{ and analogously for } G^A$$

Fourier time → energy:

$$G_E^{R,A}(\vec r, \vec r') = \langle \vec r | (E - \hat H \pm i0)^{-1} | \vec r' \rangle$$

$$\Pi(\vec r, \vec r'; t - t') \xrightarrow{\text{Fourier}} \Pi_\omega(\vec r, \vec r')$$

* Metallic phase/regime: after disorder averaging, we should get a diffusive motion

$$\left(\frac{\partial}{\partial t} - D \nabla_r^2\right) \Pi(\vec r, \vec r'; t - t') = \text{const}\, \delta(\vec r - \vec r')\, \delta(t - t')$$

$$\text{Fourier} \longrightarrow \Pi(\vec q, \omega) \sim \frac{1}{Dq^2 - i\omega}$$

* Localized phase

$$\Pi(\vec r, \vec r'; t - t') \sim e^{-|\vec r - \vec r'|/\xi}\, \theta(t)$$

$$\text{Fourier} \longrightarrow \Pi(\vec q, \omega) \sim \frac{1}{-i\omega} \frac{1}{q^2 + \xi^{-2}}$$

This is similar to correlation functions on two sides of a conventional second-order phase transition with a continuous symmetry. Indeed, consider a Heisenberg (ferromagnetic) model:

$$H = -\sum_{\langle ij \rangle} J \vec{S}_i \vec{S}_j - \vec{B} \sum \vec{S}_i \,,$$

$\langle ij \rangle$ ← nearest neighbors

$\vec{S}_i$ – n-component vector

$\vec{B}$ – external magnetic field

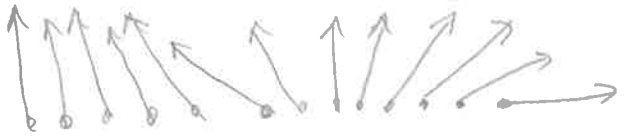
$\vec{S}_i^{\,2} = 1$

Continuous version:

$$H[\vec{S}] = \int d^d r \left[\frac{\varkappa}{2} \left(\vec{\nabla} \vec{S}(\vec{r}) \right)^2 - \vec{B} \vec{S}(\vec{r}) \right]$$

$\vec{S}(\vec{r})$ – n-component vector, $\vec{S}^{\,2}(\vec{r}) = 1$

n – component vector non-linear σ-model [NLσM]



Target manifold (to which $\vec{S}(\vec{r})$ belongs):

sphere $S^{n-1} = O(n)/O(n-1)$ symmetric space

Assume $\vec{B} \parallel 1$ axis

Independent degrees of freedom: transverse (to $\vec{B}$) part $\vec{S}_\perp$; $S_1 = \left(1 - \vec{S}_\perp^{\,2}\right)^{1/2}$

$$H[\vec{S}_\perp] = \frac{1}{2} \int d^d r \left[\varkappa \left(\nabla \vec{S}_\perp(r) \right)^2 + B \vec{S}_\perp^{\,2}(r) + O\left(\vec{S}_\perp^{\,4}(r)\right) \right]$$

* Ferromagnetic phase: broken symmetry (spontaneous magnetization at $B \to 0$)

Goldstone modes – spin waves

Correlation function (in momentum space)

$$\langle S_\perp S_\perp \rangle_q \sim \frac{1}{\varkappa q^2 + B}$$

* Paramagnetic phase: restored symmetry

Correlation length ξ

$$\langle S_\perp S_\perp \rangle_q \sim \frac{1}{q^2 + \xi^{-2}} \qquad \left[\sim e^{-|r-r'|/\xi} \text{ in } r\text{-space} \right]$$

Comparing this with $\Pi(\vec{q},\omega)$ for the localization problem, we see clear analogies

Localization transition	Ferromag.–paramagnetic phase transition
Metallic (Delocalized) phase	symmetry-broken phase (ferromagnet)
Localized phase	restored-symmetry phase (paramagnet)
Diffusion constant D	Spin stiffness $\varkappa$
Localization length ξ	Correlation length ξ
Frequency ω	external magnetic field B (explicitly breaks symmetry)

→ expect that the Anderson-localization problem can also be described by some NLσM, with some field $Q(\vec{r})$ playing a role analogous to $\vec{S}(\vec{r})$. How to derive it? What is the symmetry of the corresponding target manifold (symmetric space)?

Sketch of derivation of NLσM for Anderson localization

$$G^R_E(r_1,r_2) = \frac{-i\int D\chi D\chi^* \chi(r_1)\chi^*(r_2)\exp\left\{i\int d^dr\, \chi^*(r)[E+i\eta-H]\chi(r)\right\}}{\int D\chi D\chi^* \exp\left\{i\int d^dr\, \chi^*(r)[E+i\eta-H]\chi(r)\right\}}$$

Here $\eta \to +0$; the integration goes over fields $\chi(r)$, $\chi^*(r)$; $H = H_0 + U(r)$ — Hamiltonian, $U(r)$ — random potential. For averaging over disorder (i.e. over $U(r)$), it is convenient to get rid of the denominator (let's call it Z), which is done by means of the replica trick. One introduces n copies of the field: χ_α, $\alpha = 1 \dots n$, with the same action for each of them. The denominator then becomes $\prod_{\alpha=1}^{n} Z = Z^n$. In the end one considers the limit $n \to 0$. Using an analogous representation for G^A and multiplying them, we get

$$G^R_{E+\frac{\omega}{2}}(r_1,r_2)\, G^A_{E-\frac{\omega}{2}}(r_2,r_1) = \int D\chi D\chi^* \chi_{R,1}(r_1)\chi^*_{R,1}(r_2)\chi_{A,1}(r_2)\chi^*_{A,1}(r_1) \cdot$$

$$\boxed{\Lambda = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}_{RA}} \qquad \cdot \exp\left\{i\int d^dr\, \chi^*(r)\left[E + \left(\frac{\omega}{2}+i\eta\right)\Lambda - H\right]\chi(r)\right\}$$

where $\chi = (\chi_{R,1} \cdots \chi_{R,n}\, \chi_{A,1} \cdots \chi_{A,n})$ is a $2n$-component vector field

Assuming for convenience a Gaussian statistics of the random potential, $\langle U(\vec{r})U(\vec{r}\,')\rangle = \Gamma\,\delta(\vec{r}-\vec{r}\,')$

$$= 2\pi \underbrace{\nu}_{\text{DOS}} \underbrace{\tau}_{\text{mean-free time}}\, \delta(\vec{r}-\vec{r}\,'),$$

one can average over $U(\vec{r})$

→ quartic term in the action (i.e., in the exponent)

$$\sim \int d^d r \left(\chi^*(r)\chi(r)\right)^2 = \int d^d r\; \chi^*_\alpha(r)\chi_\alpha(r)\chi^*_\beta(r)\chi_\beta(r),$$

where α,β include R, A and replica indices.

At $\omega \to 0$, $\eta \to 0$, the action is invariant under rotations in R/A $\otimes$ replica space: $\chi \to T\chi$, $T \in U(2n)$. This group plays the role analogous to $O(n)$ in the Heisenberg model.

One can decouple this quartic term by a Gaussian integral over a $2n\times 2n$ matrix $R_{\alpha\beta}$ conjugate to $\chi_\alpha\chi^*_\beta$ (i.e. $R \sim \chi \otimes \chi^+$).

This is known as Hubbard-Stratonovich transformation.

→ integrate out χ fields

→ action in terms of R field [weight: e^{-S}]

$$S[R] = \pi\nu\tau \int d^d r\; \mathrm{tr}\, R^2 - \mathrm{tr}\ln\left[E + \left(\frac{\omega}{2} + i\eta\right)\Lambda - \hat{H}_0 - iR\right]$$

→ saddle-point approximation → equation for R:

$$R(r) = \frac{-i}{2\pi\nu\tau} \langle r | (E - \hat{H}_0 - iR)^{-1} | r\rangle \qquad (\omega \to 0)$$

→ set of solutions (saddle-point manifold)

$$R = -i\Sigma \cdot I - \frac{1}{2\tau} Q, \qquad Q = T^{-1}\Lambda T, \qquad Q^2 = 1$$

Q — $2n\times 2n$ matrix on σ-model target space M,

$Q \in M = U(2n)/U(n)\times U(n)$ symmetric Space

→ gradient expansion for a slowly varying $Q(\vec{r})$:

→ σ-model action

$$S[Q] = \frac{\pi\nu}{4}\int d^d r \,\mathrm{tr}\left[-D(\nabla Q)^2 - 2i\omega\Lambda Q\right]$$

→ correlation function $\Pi(\vec{r}_1, \vec{r}_2; \omega)$ (see p. 9.6)

$$\Pi(\vec{r}_1,\vec{r}_2;\omega) = -\int DQ\, Q^{11}_{RA}\, Q^{11}_{AR}\, e^{-S[Q]}$$

$Q^{ij}_{pp'}$ ← replica indices $1 \dots n$

pp' ← R/A - indices

Expanding Q in fluctuations $Q_\perp$ around Λ (cf. Heisenberg model, pp. 7-8), one gets

$$S[Q] = \frac{\pi\nu}{4}\int d^d r\, \mathrm{tr}\left[-D(\nabla Q_\perp)^2 + i\omega Q_\perp^2 + O(Q_\perp^3)\right],$$

$Q_\perp = -Q_\perp^+$ theory of "interacting" diffusive modes

$$\langle Q_\perp Q_\perp \rangle_{q,\omega} \sim \frac{1}{\pi\nu(Dq^2 - i\omega)}$$ — Goldstone mode = diffusion propagator

Derivation of σ-model field theory can be extended to all 10 symmetry classes. The results:

s	Ω_s: sym. space of H		M_s: sym. space of σ-model	
$0'$	A	$U(N)$	AIII	$U(2n)/U(n)\times U(n)$
$1'$	AIII	$U(p+q)/U(p)\times U(q)$	A	$U(n)$
0	AI		CII	
1	BDI		AII	
2	D		DIII	
3	DIII		D	
4	AII		BDI	
5	CII		AI	
6	C		CI	
7	CI		C	

* all 10 families of symmetric spaces appear also in M_s column!

* $M_s = \Omega_{5-s}$ (mod 8 for real)
  $M_{s'} = \Omega_{(1-s)'}$ (mod 2 for complex)

→ we should be able to understand / derive the periodic table from M_s. To understand how the topology manifests itself in the σ-model field theories, we return to the IQHE (class A, $d=2$) as a guiding example

9.4. Field theory for IQHE

Derivation of σ-model in 2D geometry in the presence of magnetic field (class A) →

$$S[Q] = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \operatorname{tr}(\nabla Q)^2 + \frac{\sigma_{xy}}{8} \operatorname{tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q - \frac{i\pi\nu}{2} \omega \operatorname{tr} Q\Lambda \right\}$$

topological term!

symmetry-breaking term

$$\boxed{\frac{\sigma_{xx}}{8} = \frac{\pi\nu D}{4}}$$

σ_{xx}, σ_{xy} – in units of $\frac{e^2}{h}$

$$Q \in M_{\text{class A}} = U(2n)/U(n) \times U(n)$$

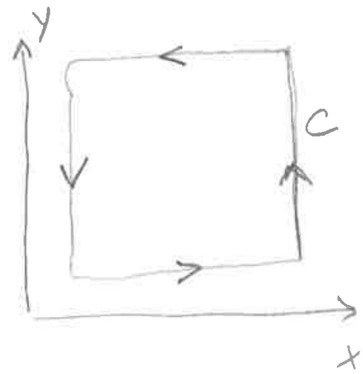
The emergence of the term $\sim \int d^2r \operatorname{tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q$ can be foreseen. It is of lowest (second) order in derivatives, and it breaks the symmetry $x \leftrightarrow y$ as the magnetic field does. Importantly, this term is topological ("θ term"). The intergrand is a full derivative:

$$\operatorname{tr} Q(\partial_x Q)(\partial_y Q) \xlongequal{Q = T\Lambda T^{-1}} 2\epsilon_{\mu\nu}\partial_\mu(\operatorname{tr} \Lambda T^{-1} \partial_\nu T)$$

⇒ this term is not seen in any order of perturbation theory and can be rewritten as a boundary integral

$$S_{top}[Q] = \int d^2r \frac{\sigma_{xy}}{8} \operatorname{tr} \epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q$$

$$= \frac{\sigma_{xy}}{2} \oint_C \operatorname{tr}(\Lambda T^{-1} \vec{\nabla} T)\, d\vec{r}$$



Consider theory on a compactified space, e.g., sphere. Boundary $C \to$ one point ("∞")
Let $Q(\infty) = -\Lambda$ (the argument does not depend on specific form of $Q(\infty)$).

$$\to \quad T = \begin{pmatrix} 0 & e^{-i\varphi} \\ e^{i\varphi} & 0 \end{pmatrix}$$ (times an overall phase that drops out)

$$\to \quad \oint_C d\vec{r}\, \mathrm{tr}\,(\Lambda T^{-1} \vec{\nabla} T) = -2i \oint_C d\vec{r}\, \vec{\nabla}\varphi =$$

$$= -4\pi i\, q, \qquad q \in \mathbb{Z}$$ – winding number

$$\to \quad \boxed{S_{top} = -2\pi i\, \sigma_{xy} \cdot q}$$ purely imaginary $q \in \mathbb{Z}$

"θ therm", $\theta = 2\pi\sigma_{xy}$

$$q\{Q\} = \frac{i}{16\pi} \int d^2r\, \mathrm{tr}\, \epsilon_{\mu\nu}\, Q\, \partial_\mu Q\, \partial_\nu Q \quad \in \mathbb{Z}$$

winding number as a 2D integral

Homotopy group:

$$\pi_2(M_{\text{class A}}) = \pi_2\left(U(2n)/U(n)\times U(n)\right) = \mathbb{Z}$$

This can be already seen on a simple example of $n=1$: $\quad U(2)/U(1)\times U(1) = S^2$ (sphere)

$\pi_2(S^2) = \mathbb{Z}$ mappings $S_2 \to S_2$

$Q \in U(2)/U(1)\times U(1) \quad \longleftrightarrow \quad$ vector $\vec{n} = (n_1, n_2, n_3)$, $\vec{n}^2 = 1$

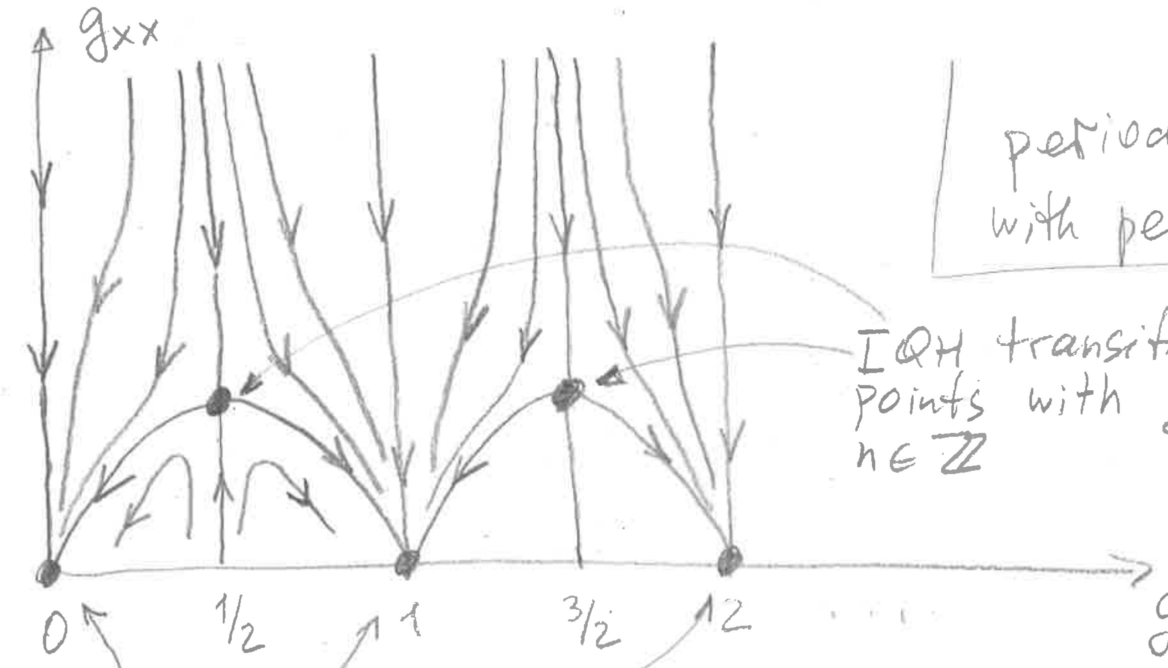
$$q = \frac{1}{8\pi} \epsilon_{\mu\nu}\, \epsilon_{abc} \int d^2r\, n_a (\partial_\mu n_b)(\partial_\nu n_c) \in \mathbb{Z}$$

Correspondence $Q \longleftrightarrow \vec{n}$:

$$Q = \begin{pmatrix} \cos\theta & i\sin\theta\, e^{i\varphi} \\ -i\sin\theta\, e^{-i\varphi} & -\cos\theta \end{pmatrix} \qquad \begin{matrix} 0 \le \theta \le \pi \\ 0 \le \varphi < 2\pi \end{matrix} \qquad Q \leftrightarrow (\theta, \varphi) \leftrightarrow \vec{n}$$

Topological term $S_{top} = -2\pi i \sigma_{xy}\, q\{Q\}$ ← integer

modifies the scaling of conductances with L (RG equations): two-parameter scaling for g_{xx}, g_{xy} ($g_{xx} = \sigma_{xx}$ for a square sample, $g_{xy} = \sigma_{xy}$)



periodic in g_{xy} with period 1

IQH transition fixed points with $g_{xy} = n + 1/2$, $n \in \mathbb{Z}$

fixed points with $g_{xy} \in \mathbb{Z}$ and $g_{xx} = 0$

$\longleftrightarrow$ $\mathbb{Z}$ topological insulator

Topology crucially affects localization properties!

Consider a system with boundary on an IQH plateau (topological insulator)

$\sigma_{xy} \equiv g_{xy} = k \in \mathbb{Z}$

Topological term $\longrightarrow$ boundary theory (1D)

$$S = \frac{k}{2} \int_C d\vec{r}\, \mathrm{tr}\,(\Lambda T^{-1} \vec{\nabla} T)$$

see p. 9.13

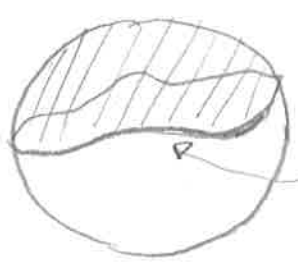
$Q = T \Lambda T^{-1}$

C $\longleftarrow$ 1D boundary

requires extension to 2D to be written in terms of Q:

$$S = \frac{k}{8} \int d^2 r\, \epsilon_{\mu\nu}\, \mathrm{tr}\, Q\, \partial_\mu Q\, \partial_\nu Q$$

Wess-Zumino term



$S = i\frac{k}{2} \cdot \text{area}$

$\longleftarrow$ defined mod 4π

$k \in \mathbb{Z} \longleftrightarrow$ number of edge modes

Emergence of the Wess-Zumino term $\longleftrightarrow$ topological protection of edge modes from localization $\longleftrightarrow$ $\mathbb{Z}$ topological insulator

9.5. From field theories of localization to the periodic table

$\circledast$ $\mathbb{Z}$ topol. insulator in d dimensions

$\longleftrightarrow$ θ - term in d dimensions possible

$\longleftrightarrow$ WZ - term in (d-1) dimensions

$\longleftrightarrow$ $\boxed{\pi_d(M_s) = \mathbb{Z}}$

This yields 3 symmetry classes that host $\mathbb{Z}$ topo insulators in each d, in exact agreement with the periodic table discussed in Chapter 8.

⊛ <u>$\mathbb{Z}_2$ topological insulator in d dimensions</u>

$\longleftrightarrow$ $\mathbb{Z}_2$ θ-term in $(d-1)$ dimensions possible

$S_{top,\mathbb{Z}_2} = i\theta q$, q defined mod 2 ,

$\theta = 0$ or π $\longleftrightarrow$ $\boxed{\pi_{d-1}(M_s) = \mathbb{Z}_2}$

$S_{top,\mathbb{Z}_2}$ with $\theta = \pi$ is similar to WZ term, makes boundary excitations non-localizable

This yields 2 classes that host $\mathbb{Z}_2$ topo insulators for each d, in agreement with periodic table from Chapter 8

Recall: $M_s = \Omega_{5-s}$ (real; mod 8) $s = 0 \ldots 7$

$M_{s'} = \Omega_{(1-s)'}$ (complex; mod 2) $s' = 0', 1'$

$$\pi_d(\Omega_s) = \pi_0(\Omega_{s+d})$$

$\rightarrow$ real: $\pi_d(M_s) = \pi_d(\Omega_{5-s}) = \pi_0(\Omega_{5-s+d})$

complex: $\pi_d(M_{s'}) = \pi_d(\Omega_{(1-s)'}) = \pi_0(\Omega_{(1-s+d)'})$.

$\rightarrow$ periodic table (entries depend only on $s-d$)

	Hamiltonian	σ-model	Top. insulators				
S	Ω_S	M_S	$d=0$	$d=1$	$d=2$	$d=3$	$d=4$
0	AI	CII	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
1	BDI	AII	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
2	D	DIII	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
3	DIII	D	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
4	AII	BDI	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
5	CII	AI	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
6	C	CI	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
7	CI	C	0	0	0	$\mathbb{Z}$	0
0′	A	AIII	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
1′	AIII	A	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0