10. Fractional quantum Hall effect

Topological phases of matter that we considered up to now in this course are termed symmetry-protected topological phases.

They can be realized with non-interacting/mean-field fermionic Hamiltonians. They exist also in the presence of interaction but the interacting states can be continuously connected to non-interacting ones.

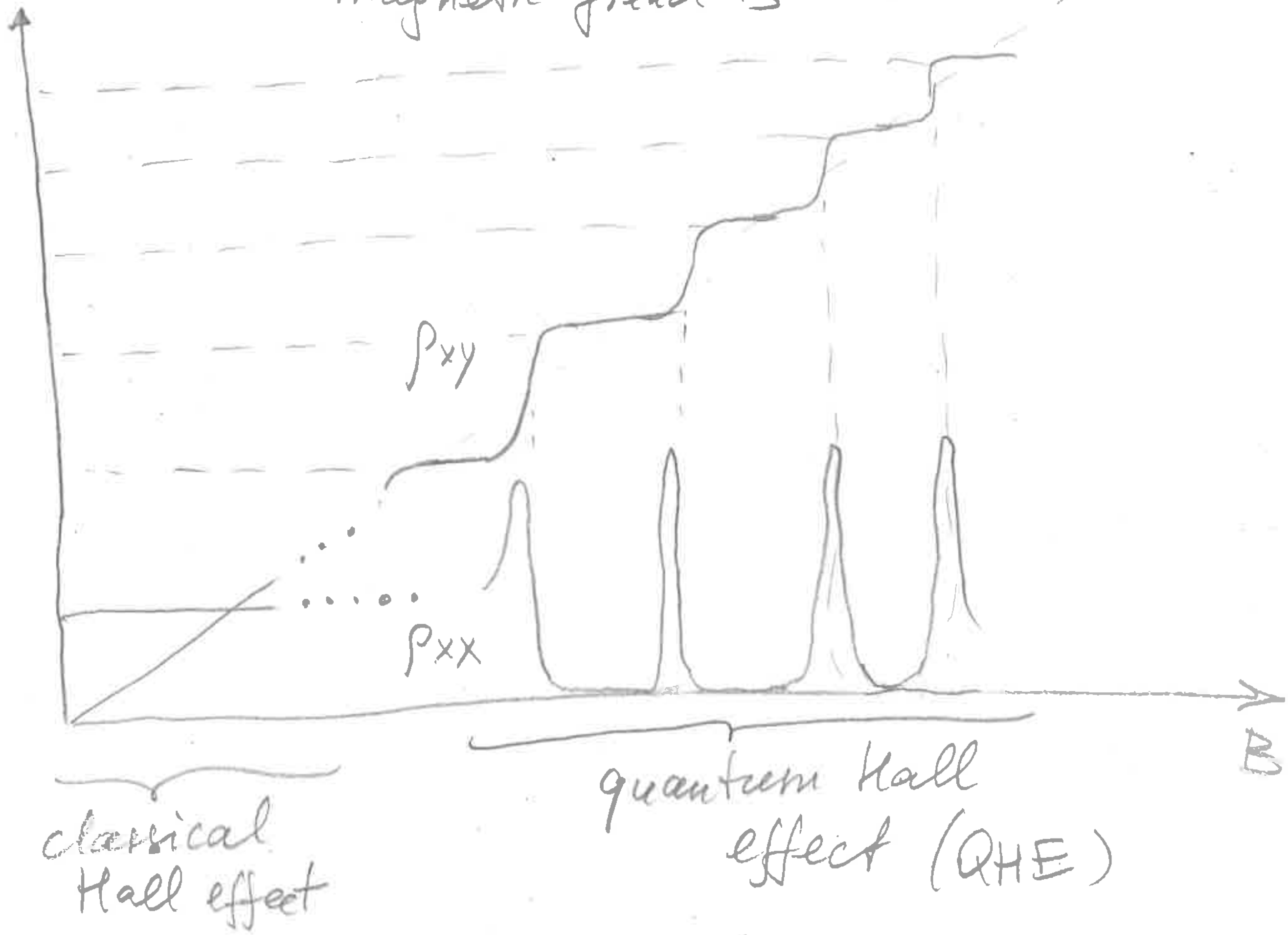
Now we consider a different class of states of matter: phases of matter exhibiting topological order (the term coined by Wen in 1989). In these states of matter, interaction leads to dramatic changes of physical properties. Paradigmatic example: FQHE (fractional quantum Hall effect). Also: spin liquids

Key manifestations of topological order:

* ground-state degeneracy that depends on topology (genus) of the real-space manifold (torus, ...)
* fractional statistics of excitations
* fractional charge of excitations
* long-range entanglement

Topological order is distinct from the more conventional order (Landau theory) associated with spontaneous symmetry breaking and long-range correlations of local observables.

2D electron gas in transverse magnetic field B



QHE plateaus: $\begin{cases} \rho_{xx} = 0 \\ \rho_{xy} = \dfrac{h}{\nu e^2} \end{cases}$

$\nu = 1, 2, 3, \dots \; ; \quad \nu = \frac{1}{3}, \frac{1}{5}, \frac{2}{3}, \frac{2}{5}, \frac{3}{5}, \dots = \frac{p}{q}$

$= n \in \mathbb{Z}$ fractional (FQHE)

integer (IQHE)

Plateau with $\rho_{xy} = \frac{h}{\nu e^2}$ is centered at the filling factor $\frac{hc}{e}\frac{n_e}{B} = 2\pi l_0^2 \cdot n_e = \frac{N_e}{N_\Phi} = \nu$

$l_0 = \left(\frac{\hbar c}{eB}\right)^{1/2}$ — mag. length,

N_e — # of electrons, N_Φ — # of flux quanta

FQHE: experimental discovery: Tsui, Störmer 1982
theoretical explanation: Laughlin (1983)
Nobel Prize: Tsui, Störmer, Laughlin (1998).

10.1. Laughlin wave functions

$$H = -\frac{1}{2m}\left(\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2 \quad ; \quad \text{charge } -e \quad (e>0)$$

It is convenient to chose $\vec{B} = -B\hat{z}$ $(B>0)$

$$(A_x, A_y) = \frac{-B}{2}(-y, x) = \frac{B}{2}(y, -x)$$

Compex coordinates: $z = x+iy$, $z^* = x-iy$

$$x = \frac{1}{2}(z+z^*) \qquad \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$$

$$y = \frac{1}{2i}(z-z^*) \qquad \partial_{z^*} = \frac{1}{2}(\partial_x + i\partial_y)$$

$$H = -\frac{1}{m}\left(D_z D_{z^*} + D_{z^*} D_z\right)$$

$$D_z = \partial_z + ieA_z \qquad A_z = \frac{1}{2}(A_x - iA_y) = -\frac{B}{4i}z^*$$

$$D_{z^*} = \partial_{z^*} + ieA_{z^*} \qquad A_{z^*} = \frac{1}{2}(A_x + iA_y) = \frac{B}{4i}z$$

One can rewrite H as follows $(\omega_c = eB/mc)$

$$H = e^{-|z|^2/4\ell_0^2}\left[-\frac{2}{m}\left(\partial_z - \frac{1}{2\ell_0^2}z^*\right)\partial_{z^*}\right]e^{|z|^2/4\ell_0^2} + \frac{\hbar\omega_c}{2}$$

$\ell_0 = \left(\frac{\hbar c}{eB}\right)^{1/2}$ – magnetic length, $\quad |z|^2 = zz^*$

This allows one to immediately write down eigenfunctions and energies of H:

Lowest Landau level (LL):

$$\psi^{(0)}(z, z^*) = e^{-|z|^2/4\ell_0^2} f(z) \qquad E^{(0)} = \frac{1}{2}\hbar\omega_c$$

$f(z)$ — any analytic function of z

$$\psi^{(1)}(z,z^*) = e^{-|z|^2/4\ell_0^2}\left[\partial_z - \frac{1}{2\ell_0^2}z^*\right]f(z),$$

first excited LL

$$E^{(1)} = \frac{3}{2}\hbar\omega_c$$

$$\psi^{(n)}(z,z^*) = e^{-|z|^2/4\ell_0^2}\left[\partial_z - \frac{1}{2\ell_0^2}z^*\right]^n f(z), \quad E^{(n)} = \left(n+\frac{1}{2}\right)\hbar\omega_c$$

n-th LL

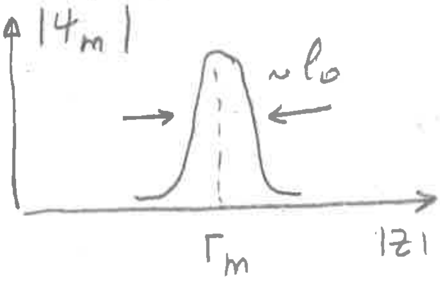
To study FQHE, we will focus on the lowest (zeroth) LL. As a basis, we choose states that carry a definite angular momentum $m\hbar$:

$$\psi_m^{(0)}(z,z^*) = \frac{1}{\sqrt{2\pi\ell_0^2\cdot 2^m m!}}\, z^m e^{-\frac{1}{4}|z|^2/\ell_0^2}$$

We omit the superscript (0) below, since we consider only the lowest LL.

The wave function ψ_m has a ring shape, with a maximum at $|z| = r_m = \sqrt{2m}\cdot\ell_0$

$\pi r_m^2 = 2\pi\ell_0^2\cdot m \Rightarrow$ one state per flux quantum



These are single-particle states.

We want to write down a many-body wave function.

Consider first $\nu = 1$: filled LL.

$$\Psi_1(z_1 \dots z_N) = \mathcal{A}\left[(z_1)^0 (z_2)^1 \dots (z_N)^{N-1}\right]\cdot e^{-\sum_i |z_i|^2/4\ell_0^2}$$

antisymmetrization

* Here we ignored normalization factor for brevity.

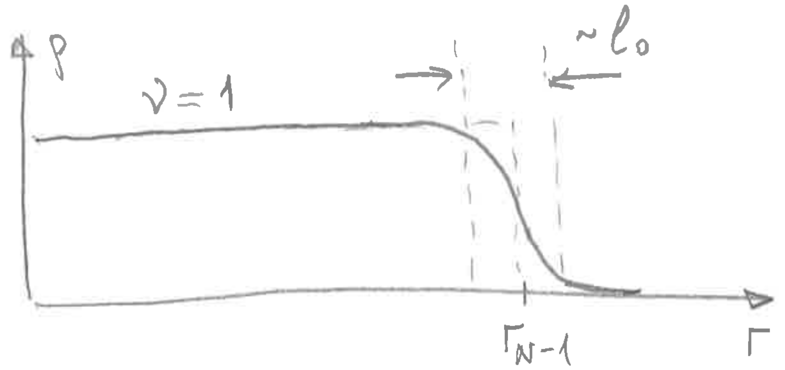
* in fact, in previous notations, $\Psi_1 = \Psi_1(z_1 \dots z_N, z_1^* \dots z_N^*)$ skip for brevity

$$\Psi_1(z_1 \dots z_N) = \det \begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ - & - & - & - \\ - & - & - & - \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix} \cdot e^{-\sum_i |z_i|^2/4l_0^2}$$

This Slater determinant has a form of the Vandermonde determinant that can be identically expressed as a Vandermonde polynomial:

$$\Psi_1(z_1 \dots z_N) = \prod_{i<j} (z_i - z_j)\, e^{-\sum_i |z_i|^2/4l_0^2}$$

The corresponding density profile is uniform (filled LL) within the disk of radius $r_{N-1} \approx r_N$



This follows from the fact that Ψ_1 describes a totally filled LL. Follows also from the plasma analogy, see below.

Now we are prepared to formulate the famous Laughlin wave function for $\nu = 1/3$:

$$\Psi_{1/3}(z_1 \dots z_N) = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4l_0^2}$$

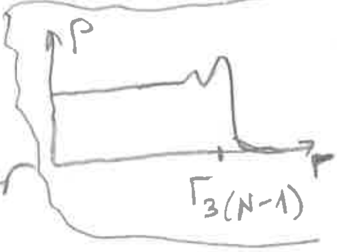
* it is in the lowest LL since is analytic function of z_i
* has a uniform density corresponding to $\nu = 1/3$ (proof below)

* has a third-order zero $(z_i - z_j)^3$ for any pair of electrons. This is favorable for a repulsive interaction

It is obvious that $\Psi_{1/3}$ describes a circularly-symmetric droplet. Assume that it is uniform (proof below) and find its radius.

Highest power for each z_i: $m_{max} = 3(N-1) \approx 3N$

$\longrightarrow$ radius $r_{m_{max}} = \sqrt{6N} \cdot \ell_0$



$$\boxed{\nu = \frac{hc}{eB} \frac{N}{\pi r^2_{m_{max}}} = \frac{1}{3}}$$ filling factor

Proof that the droplet is uniform: plasma analogy

$$|\Psi_{1/q}(z_1 \ldots z_N)|^2 = e^{-\beta V(z_1 \ldots z_N)}, \quad \beta = 2q$$

for generality, for $\Psi_{1/q} = \prod_{i<j} (z_i - z_j)^q e^{-\sum_i |z_i|^2/\ell_0^2}$, $q = 1, 3, 5, \ldots$

$$V(z_1 \ldots z_N) = -\sum_{i<j} \ln|z_i - z_j| + \frac{1}{4q\ell_0^2} \sum_i |z_i|^2$$

$\rightarrow$ classical plasma of particles with charge 1 and 2D Coulomb potential $-\ln|z_i - z_j|$.

The second term: potential of a uniform background with density $\rho = 1/2\pi\ell_0^2 q$. Indeed,

$$\nabla^2 \left(\frac{1}{4q\ell_0^2} |z|^2 \right) = \frac{1}{q\ell_0^2} \xrightarrow[\text{Poisson eq. in 2D}]{\nabla^2 \varphi = -2\pi\rho} \rho = \frac{-1}{2\pi q \ell_0^2}$$

Plasma screens completely the background density

$\rightarrow$ uniform droplet with $\rho = \frac{1}{2\pi q \ell_0^2} \Longrightarrow \boxed{\nu = 1/q}$

Laughlin wave function is only an approximation for a true ground state for electrons with a Coulomb interaction. However, it properly captures all key properties of the ground state. [Note also that $\Psi_{1/3}$ is an exact ground state for the system with interaction $\propto \partial_r^2 \delta(\vec{r})$.]

10.2. Quasiparticle and quasihole excitations of Laughlin states: Charge fractionalization

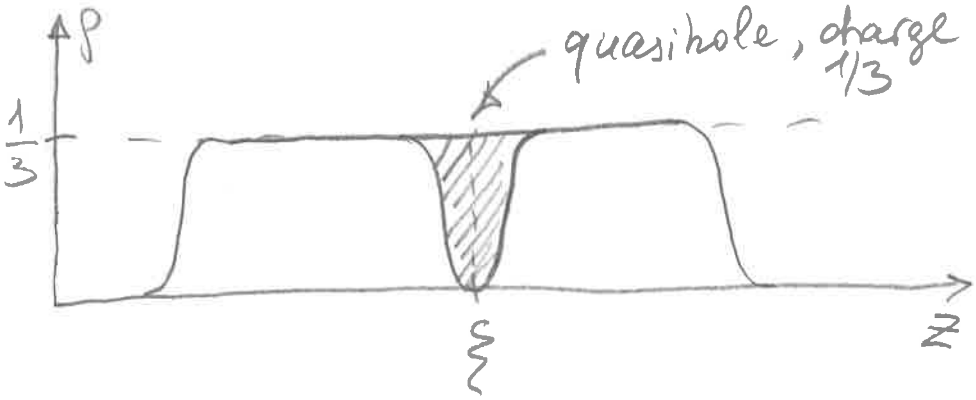
A quasihole in the Laughlin state $\Psi_{1/3}$ is described by the following many-body wave function

$$\Psi^h_{1/3}(z_1 \dots z_N; \xi) = \left[C(\xi, \xi^*)\right]^{-\frac{1}{2}} \cdot \left[\prod_i (z_i - \xi)\right] \Psi_{1/3}(z_1 \dots z_N)$$

position of quasihole

normalization factor

Within the plasma analogy the factor corresponds to an additional contribution $-\frac{1}{3} \ln|z_i - \xi|$ to the potential $V(z_1 \dots z_N)$, which is an interaction with an additional charge 1/3 at point ξ. The Coulomb plasma screens it $\Rightarrow$ quasihole has a charge $+\frac{e}{3}$ (if the electron has charge $-e$)



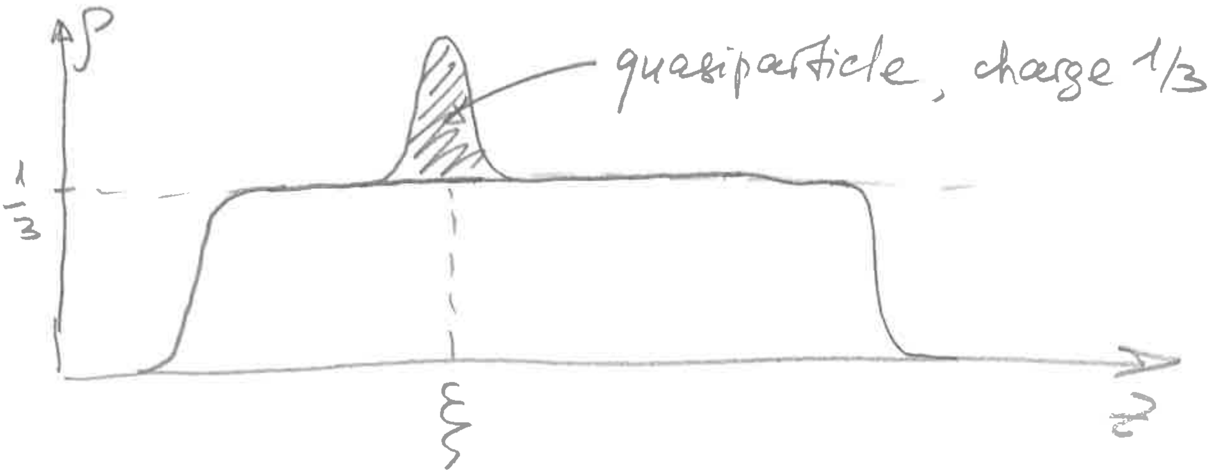
Extension to $\Psi_{1/q}$ Laughlin state: quasihole has a charge $+\frac{e}{q}$

$q = 3, 5, \dots$

Quasiparticle in Laughlin state

$$\Psi^{p}_{1/3}(z_1 \ldots z_N; \xi) = \left[\prod_i \left(2\frac{\partial}{\partial z_i} - \frac{1}{\ell_0^2}\xi^* \right) \prod_{i<j} (z_i - z_j)^3 \right] \cdot$$

$$\cdot\, e^{-\sum_i |z_i|^2 / 4\ell_0^2}$$



To understand better the effect of $\prod_i (z_i - \xi)$ and $\prod_i \left(2\frac{\partial}{\partial z_i} - \xi^*\right)$, consider $\xi = 0$. Then $\prod_i z_i$ increases the angular momentum of each single-particle state: $z_i^m \mapsto z_i^{m+1}$, i.e. $m \mapsto m+1$, while $\frac{\partial}{\partial z_i}$ decreases the angular momentum: $m \mapsto m-1$.

The increase $m \mapsto m+1$ repell particles from $z=0$, creating a quasihole, while the decrease $m \mapsto m-1$ moves electrons closer to $z=0$, creating a quasiparticle.

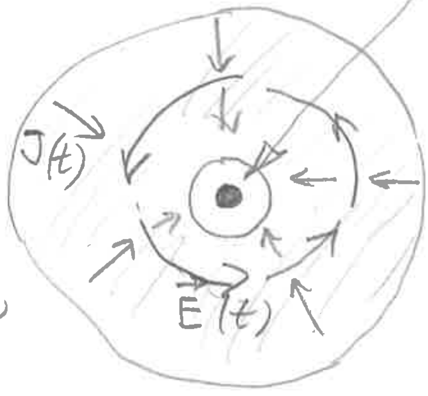
For a clean system, there will be an <u>energy gap</u> for creating quasiparticle / quasihole excitations in a Laughlin state, similarly to an energy gap between LLs for IQHE.

In the presence of disorder, quasiparticles or quasiholes will be Anderson-localized when the filling factor ν will deviate from $1/9$

$\rightarrow$ plateaus in $\sigma_{xy} = \frac{1}{9}\frac{e^2}{h}$: FQHE

Relation between fractional charge and fractional Hall conductivity

We now inspect the flux-insertion argument that was used to explain integer quantization of σ_{xy} for IQHE (Sec. 4.2.). Make an opening in a sample and thread with a flux $\Phi(t)$. Change $\Phi(t)$ adiabatically from 0 to flux quantum Φ_0.



$\rightarrow$ Faraday electric field $\vec{E}(t)$,

$$\oint \vec{E}\, d\vec{l} = -\frac{1}{c}\frac{d\Phi(t)}{dt}$$

$\rightarrow$ charge transported from the external edge to the internal edge (i.e. to the orifice):

$$Q = \sigma_{xy}\int dt \oint \vec{E}\, d\vec{l} = -\sigma_{xy}\frac{\Phi_0}{c} = -\frac{e}{9},$$

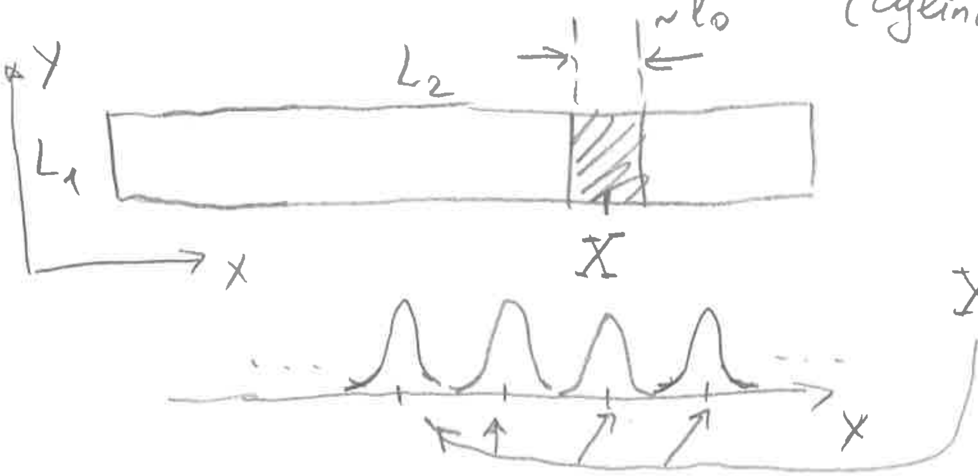
i.e. exactly the fractional charge of a quasiparticle

$\sigma_{xy} = \frac{1}{9}\frac{e^2}{h}$

$\Rightarrow$ for FQHE, edge modes are "formed" by fractional quasiparticles: such quasiparticles can be transported from one edge to the other one through the 2D bulk!

Furthermore, the ground state of the bulk should have changed upon insertion of Φ_0, even though the Hamiltonian returned to its original form – <u>topological ground-state degeneracy</u>. This is easy to see in the Tao-Thouless limit. (A more general discussion comes below.)

Tao-Thouless limit: $L_1 \times L_2$ sample, $L_1 \to 0$ (finite but small)
(cylinder)



$$k_y = \frac{2\pi}{L_1} n_y, \quad n_y \in \mathbb{Z}$$

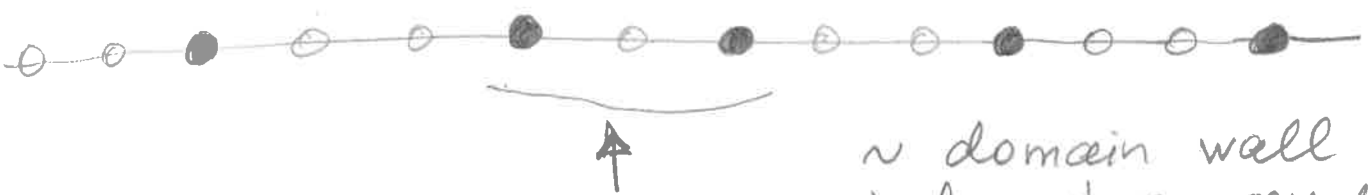
$$X = -k_y l_0^2$$

→ 1D lattice model

→ ground state for $\nu = 1/3$:

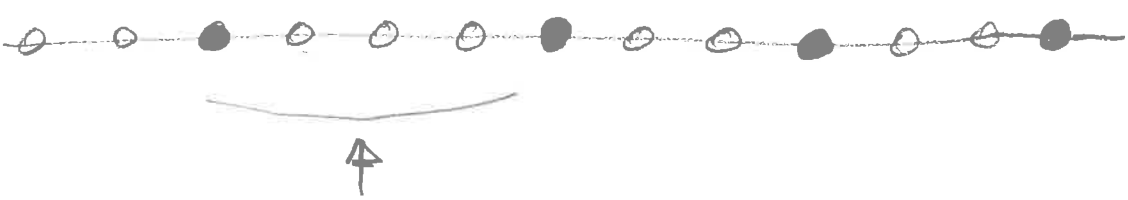
3-fold degenerate

quasiparticle with charge 1/3:



~ domain wall between two ground states

quasihole:



Transporting a quasiparticle (or quasihole) changes the ground state. After 3 quasiparticles are transported, the original ground state gets restored

10.3. Possibility of fractional statistics in 2D geometry: Anyons

Leinaas, Myrheim 1977; Wilczek 1982

Consider a wave function of identical particles $\psi(\vec{r}_1, \vec{r}_2 \dots \vec{r}_N)$. We will assume hard-core constraint: two particles cannot be at the same point. Usual argument for possible statistics: exchange coordinates of two particles:

$$P_{12}\psi(\vec{r}_1\vec{r}_2\dots\vec{r}_N) = \psi(\vec{r}_2\vec{r}_1\dots\vec{r}_N) = \psi'(\vec{r}_1\vec{r}_2\dots\vec{r}_N).$$

This is the same configuration for identical particles $\Longrightarrow$ $\psi'(\vec{r}_1\vec{r}_2\dots\vec{r}_N) = e^{i\theta}\psi(\vec{r}_1\vec{r}_2\dots\vec{r}_N)$.

Repeating this twice should return ψ back:

$$P_{12}^2\psi = P_{12}\psi' = \psi \Rightarrow P_{12}^2 = 1 \Rightarrow P_{12} = \pm 1.$$

This leads to usual two possibilities:

$P_{12} = +1$, i.e., $\theta = 0$ bosons

$P_{12} = -1$, i.e., $\theta = \pi$ fermions

It turns out that this is not the whole story about the statistical phase. One may have (in addition) a <u>topology-related phase</u> when physically exchanging position of two particles. To understand this, it is sufficient to inspect the case of 2 particles. Assume d-dimensional space

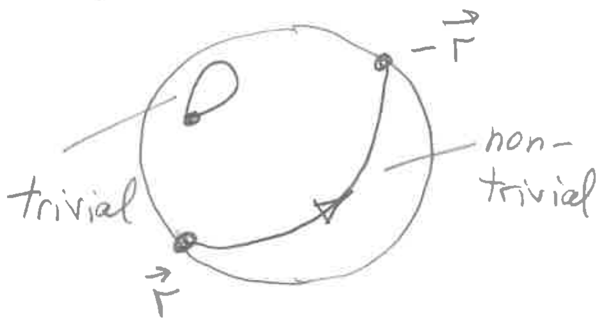
$$\vec{r}_1, \vec{r}_2 \longrightarrow \begin{cases} \vec{R} = (\vec{r}_1 + \vec{r}_2)/2 & \text{(center of mass) - ignore} \\ \vec{r} = \vec{r}_1 - \vec{r}_2 & \text{— exchange involves only } \vec{r} \end{cases}$$

Exchange: $\vec{r} \to -\vec{r}$.

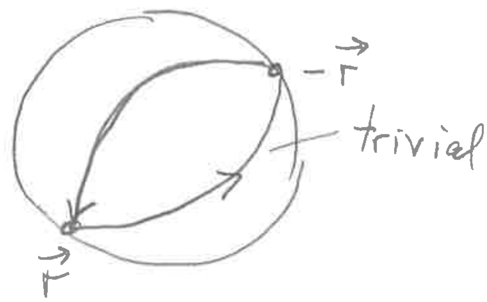
The point $\vec{r} = 0$ is excluded

Preserving the topology, we can fix $|\vec{r}| = r$, so that we get a sphere S^{d-1}. Its opposite points are identified, since $\vec{r}$ and $-\vec{r}$ correspond to the same configuration (identical particles!) In 3D, we have a sphere S^2 with identified opposite points. The exchange $\vec{r} \to -\vec{r}$ corresponds to a loop on this manifold, which is topologically non-trivial:

So, there is a factor $e^{i\theta}$ associated with this process.



At the same time, repeating this process twice, we get a trivial path (topologically):

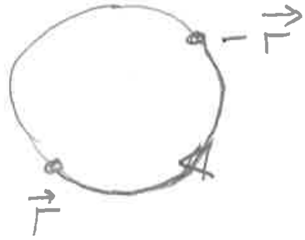


This is because π_1 homotopy group of this manifold is $\mathbb{Z}_2$.

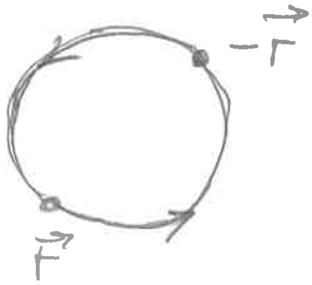
$$\Rightarrow (e^{i\theta})^2 = 1 \quad \Longrightarrow \quad \theta = 0 \text{ or } \pi \text{ — bosons or fermions.}$$

Now consider $\underline{d=2}$.

We have now a sphere S^1 (a circle) with identified opposite points.



Now not only a single-exchange process non-trivial topologically, but also a double exchange, and also a path with n exchanges, $n \in \mathbb{Z}$. The homotopy group π_1 is now $= \mathbb{Z}$.



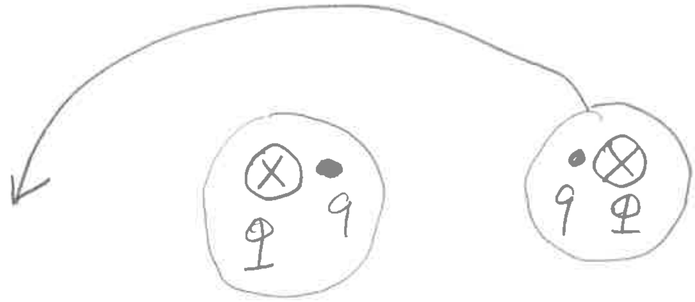
$\Rightarrow$ An exchange 

may induce a phase factor $e^{i\theta}$ with any θ.

Particles with such unconventional statistics (in 2D) are called <u>anyons</u>.

How can an anyon emerge in quantum mechanics if the original microscopic model does not contain anyons? If we recall a discussion of the Aharonov-Bohm effect as Berry phase (Sec. 2.2), we can understand that anyons can emerge as <u>bound states of charge and flux.</u> For example, consider bound states of a boson (with charge q) and a flux Φ. Exchanging two of such bound states — or, equivalently, moving one of them half-way around the other one — we will get a phase factor $e^{i\theta}$, where

$$\theta = \pi \frac{q}{e}\frac{\Phi}{\Phi_0} + \pi \frac{q}{e}\frac{\Phi}{\Phi_0} = 2\pi \frac{q}{e}\frac{\Phi}{\Phi_0}$$



Here one term comes from charge going half-way around flux, and the second term from flux going half-way around the charge

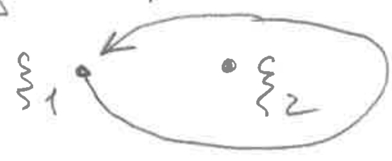
Thus, charge-flux bound states are in general anyons.

We will now show that excitations in FQH systems are anyons

10.4. Fractional statistics of FQH excitations

Consider the Laughlin wave function $\Psi_{1/3}$. One can guess the anyonic statistics of quasiparticles / quasiholes as follows. The factor $(z_i - z_j)^3$ for any pairs of electrons acquires the phase 3π when they are exchanged. Now, each electron can be viewed as a conglomerate of 3 quasiparticles. This suggest the phase $3\pi/3^2 = \pi/3$ for an exchange of two quasiparticles. And similarly, the phase π/m for an exchange of two quasiparticles in a $\Psi_{1/m}$ Laughlin state $(\nu = 1/m)$. This guess is correct. We present now its accurate derivation.

To do this, we will calculate the Berry phase for moving a quasihole ξ_1 around a quasihole ξ_2.



Consider first moving one quasihole

$$\Psi^h_{1/m}(z_1 \dots z_N; \xi) = \left[C(\xi,\xi^*)\right]^{-1/2} \cdot \underbrace{\left[\prod_i (z_i - \xi)\right] \Psi_{1/m}(z_1 \dots z_N)}_{\tilde{\Psi}^h_{1/m}(z_1 \dots z_N; \xi)}$$

$$C(\xi,\xi^*) = \langle \tilde{\Psi}^h_{1/m}(\xi^*) | \tilde{\Psi}^h_{1/m}(\xi) \rangle$$

analytic function of ξ^* ← analytic function of ξ

Components of the Berry connection

$$A_\xi = i \langle \Psi^h_{1/m} | \partial_\xi | \Psi^h_{1/m} \rangle$$

$$A_{\xi^*} = i \langle \Psi^h_{1/m} | \partial_{\xi^*} | \Psi^h_{1/m} \rangle$$

Since $\tilde{\Psi}^h_{1/m}$ is an analytic function of ξ, we find

$$A_{\xi^*} = i\, C^{1/2}(\xi,\xi^*) \frac{\partial}{\partial \xi^*} C^{-1/2}(\xi,\xi^*).$$

To find A_ξ, we integrate by part,

$A_\xi = -i \left[\partial_\xi \langle \Psi^h_{1/m} |\right] | \Psi^h_{1/m} \rangle$, and note that

$\langle \tilde{\Psi}^h_{1/m} |$ is an analytic function of ξ^*

$$\Rightarrow A_\xi = -i\, C^{1/2}(\xi,\xi^*) \frac{\partial}{\partial \xi} C^{-1/2}(\xi,\xi^*).$$

To determine $C(\xi,\xi^*)$, we use the plasma analogy.

Note that

$$\int d^2 z_i \left| e^{-|\xi|^2/4m\ell_0^2} \prod_i (z_i - \xi) \prod_{i<j} (z_i - z_j)^m \prod_i e^{-|z_i|^2/4\ell_0^2} \right|^2 \equiv e^{-\beta V(\xi,\xi^*)}$$

is the partition function of the plasma with a test charge $1/m$ placed at ξ.

Crucially, we have included here the factor $e^{-|\xi|^2/4m\ell_0^2}$ which corresponds to the interaction of test charge with the background charge.

Thus, $V(\xi, \xi^*)$ is the free energy of the test charge.

Since plasma screens potentials, $V(\xi, \xi^*)$ should be independent on the position of test charge, $V(\xi, \xi^*) = \text{const}$. Thus,

$$C^{-1/2}(\xi, \xi^*) = e^{-|\xi|^2/4m\ell_0^2}$$

up to a constant that does not depend on (ξ, ξ^*) and thus plays no role here

$$\Rightarrow \quad A_\xi = \frac{i\xi^*}{4m\ell_0^2}, \quad A_{\xi^*} = \frac{-i\xi}{4m\ell_0^2}$$

$\xi = x + iy$, $\xi^* = x - iy$

$A_\xi = (A_x - iA_y)/2$ (like ∂_ξ)

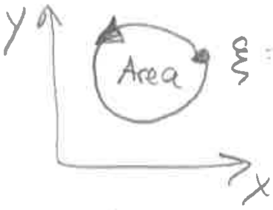
$A_{\xi^*} = (A_x + iA_y)/2$ (like ∂_{ξ^*})

$$\rightarrow \quad A_x = A_\xi + A_{\xi^*} = \frac{y}{2m\ell_0^2}, \quad A_y = i(A_\xi - A_{\xi^*}) = -\frac{x}{2m\ell_0^2}$$

This is exactly the vector potential of a uniform field (Berry curvature)

$$b = \partial_x A_y - \partial_y A_x = -\frac{1}{m\ell_0^2}$$

When the quasihole is moved around a loop, it acquires the Berry phase



$$\gamma = \oint \vec{A}\, d\vec{l} = b \cdot \text{Area} = -\frac{1}{m}\frac{\text{Area}}{\ell_0^2} =$$

$$= \frac{2\pi}{m}\frac{\Phi}{\Phi_0}$$

, where Φ is the flux of physical magnetic field through the loop. (We recall that $\vec{B} = -B\hat{z}$, with $B > 0$)

This implies the charge $e^* = \frac{e}{m}$ of the quasihole, in full consistency with what we found earlier (Recall that $e>0$, electron charge is $-e$.)

Now we apply this logics to two quasiholes. The free energy $V(\xi_1, \xi_1^*, \xi_2, \xi_2^*)$ is now given by

$$e^{-\beta V(\xi_1,\xi_1^*,\xi_2,\xi_2^*)} = \int \prod_i d^2 z_i \Big| e^{-(|\xi_1|^2+|\xi_2|^2)/4m l_0^2} \cdot$$

$$\cdot \left|\xi_1 - \xi_2\right|^{1/m} \prod_{\substack{i \\ a=1,2}} (\xi_a - z_i) \prod_{i<j} (z_i - z_j)^m \prod_i e^{-|z_i|^2/4l_0^2} \Big|^2 ,$$

where we also included interaction between two test charges (at ξ_1 and ξ_2). Again, $V(\xi_1, \xi_1^*, \xi_2, \xi_2^*)$ should be independent on ξ_1, ξ_2:

$V(\xi_1, \xi_1^*, \xi_2, \xi_2^*) = \text{const}$. Thus, the 2-quasihole wave function is (up to a constant factor of no interest)

$$\Psi^{2h}_{1/m}(z_1 \dots z_N; \xi_1, \xi_2) = (\xi_1 - \xi_2)^{\frac{1}{2m}} (\xi_1^* - \xi_2^*)^{\frac{1}{2m}} e^{-\frac{|\xi_1|^2 + |\xi_2|^2}{4m l_0^2}} \cdot$$

$$\cdot \prod_i \left[(\xi_1 - z_i)(\xi_2 - z_i)\right] \cdot \Psi_{1/m}(z_1 \dots z_N)$$

We want to calculate the Berry phase for the process  ξ_1 Set $\xi_2 = 0$, $\xi_1 = \xi$.

Calculating the Berry connection, we obtain

$$A_\xi = \frac{i\xi^*}{4m l_0^2} - \frac{i}{2m\xi}, \qquad A_{\xi^*} = \frac{-i\xi}{4m l_0^2} + \frac{i}{2m\xi^*}$$

The first term in A_ξ and A_{ξ^*} is the same as in the absence of the second quasihole (ξ_2). It corresponds to the quasihole ξ_1 moving in a uniform field. The second term is related to mutual statistics of the quasiholes. It originates from

$$\xi^{-\frac{1}{2m}}\,\partial_\xi\,\xi^{\frac{1}{2m}} = \frac{1}{2m\xi}$$

$$(\xi^*)^{-\frac{1}{2m}}\,\partial_{\xi^*}(\xi^*)^{1/2m} = \frac{1}{2m\xi^*}$$

This contribution yields $\left(r^2 = x^2+y^2 = |\xi|^2\right)$

$$A_x = A_\xi + A_\xi^* = -\frac{1}{m}\frac{y}{r^2}\,, \qquad A_y = i\left(A_\xi - A_{\xi^*}\right) = \frac{1}{m}\frac{x}{r^2}$$

This is a vector potential of a flux tube at the origin. The Berry phase is

$$\gamma = \oint \vec{A}\,d\vec{l} = \frac{2\pi}{m}$$

The phase θ corresponding to exchange of two quasiholes is given by $\frac{1}{2}\gamma$

$$\boxed{\theta = \frac{\pi}{m}}$$ fractional statistics (anyons)

10.5. Topological degeneracy of FQH ground state

In Sec. 4.4, we defined magnetic translations for a single particle in a uniform magnetic field

$$H = \frac{1}{2m}\left[\vec{p} + \frac{e}{c}\vec{A}(\vec{r})\right]^2 \qquad \vec{\nabla}\times\vec{A}(\vec{r}) = B$$

see also discussion of Tao-Thouless limit, p. 10.10

$T_{\vec a}$ – conventional translation

$$T_{\vec a} H T_{\vec a}^{+} = \frac{1}{2m}\left[\vec p + \frac{e}{c}\vec A(\vec r + \vec a)\right]^2 = H' \neq H$$

$$\vec\nabla \times \vec A(\vec r) = \vec B = \vec \nabla \times \vec A(\vec r + \vec a) \Rightarrow A(\vec r) \text{ and } \vec A(\vec r + \vec a)$$
related by a gauge transf.

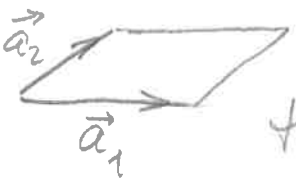
$$\Rightarrow \vec A(\vec r + \vec a) = \vec A(\vec r) + \vec\nabla f_{\vec a}(\vec r)$$

$$\Rightarrow \quad H = e^{i\varphi_{\vec a}(\vec r)} H' e^{-i\varphi_{\vec a}(\vec r)} \qquad \varphi_{\vec a}(\vec r) = \frac{e}{\hbar c} f_{\vec a}(\vec r)$$

$$\Rightarrow \boxed{\tilde T_{\vec a} H \tilde T_{\vec a}^{+} = H}, \qquad \tilde T_{\vec a} = e^{i\varphi_{\vec a}(\vec r)} T_{\vec a}$$

magnetic translation operator

$$\tilde T_{\vec a_1} \tilde T_{\vec a_2} = e^{2\pi i \frac{\Phi_{\vec a_1 \vec a_2}}{\Phi_0}} \cdot \tilde T_{\vec a_2} \tilde T_{\vec a_1}$$



$\Phi_{\vec a_1 \vec a_2}$ – flux through the parallelogram

$$[\tilde T_{\vec a_1}, \tilde T_{\vec a_2}] = 0 \quad \text{if} \quad \Phi_{\vec a_1 \vec a_2}/\Phi_0 \in \mathbb{Z}$$

For a many-body system of N particles ($j = 1 \dots N_e$) define the magnetic translation operator as

$$\tilde T_{\vec a} = \prod_{j=1}^{N_e} T_{\vec a}^{(j)}$$

← acts on coordinates of j-th particle

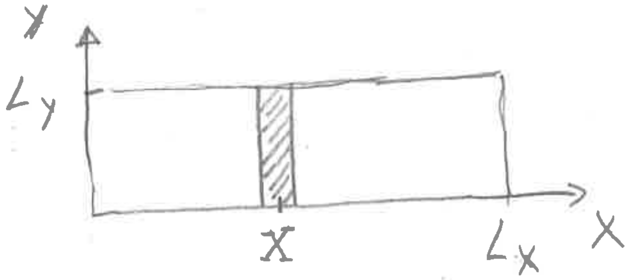
Consider a system on a torus $L_x \times L_y$ (rectangle with periodic boundary conditions). Assume an integer number N_φ of flux quanta:

$$N_\varphi = L_x L_y / 2\pi l_0^2 \in \mathbb{Z}$$

What are allowed translations in x and y directions on a torus (i.e. those that preserve periodic boundary conditions)? These are

$$\tilde{T}_{L_x \cdot \hat{x}/N_\varphi} \equiv t_x \quad \text{and} \quad \tilde{T}_{L_y \cdot \hat{y}/N_\varphi} \equiv t_y$$

To see this, consider Landau gauge $A_x = 0$, $A_y = Bx$



$$\Psi_X(x,y) = e^{ik_y y} e^{-(x-X)/2l_0^2}$$

$$X = -k_y l_0^2$$

$$k_y = -2\pi n_y / L_y \ , \quad n_y \in \mathbb{Z}$$

$$0 \leq n_y < \frac{L_x L_y}{2\pi l_0^2} = N_\varphi$$

Translations in $\hat{x}$ direction that are consistent with periodic boundary condition (i.e., with quantization of k_y) are multiples of $\Delta x = l_0^2 \Delta k_y = l_0^2 \cdot \frac{2\pi}{L_y} = \frac{L_x}{N_\varphi}$, which is exactly t_x stated above.

We have $t_x t_y = e^{i\theta} t_y t_x$,

where $\theta = 2\pi \Phi_{L_x \cdot \hat{x}/N_\varphi,\, L_y \cdot \hat{y}/N_\varphi} \cdot N_e = 2\pi N_\varphi \frac{1}{N_\varphi^2} N_e$

flux through the whole system $L_x \times L_y$

contributions of N_e electrons to θ add

$\Rightarrow$

$$\theta = 2\pi \frac{N_e}{N_\varphi} = 2\pi\nu = \frac{2\pi}{m} \quad \text{for } 1/m \text{ Laughlin state}$$

Choose ground state(s) to be eigenfunctions of t_y

$$t_y \Psi_0 = e^{i\lambda} \Psi_0$$

Consider $\Psi_1 = t_x \Psi_0$

$$t_y \Psi_1 = t_y t_x \Psi_0 = e^{-i\frac{2\pi}{m}} t_x t_y \Psi_0 = e^{-i\frac{2\pi}{m}+i\lambda} \underbrace{t_x \Psi_0}_{=\Psi_1}$$

Similarly, $\Psi_2 = t_x \Psi_1 = t_x^2 \Psi_2$

$$\rightarrow \quad t_y \Psi_2 = e^{-2\cdot\frac{2\pi i}{m}} \Psi_2$$

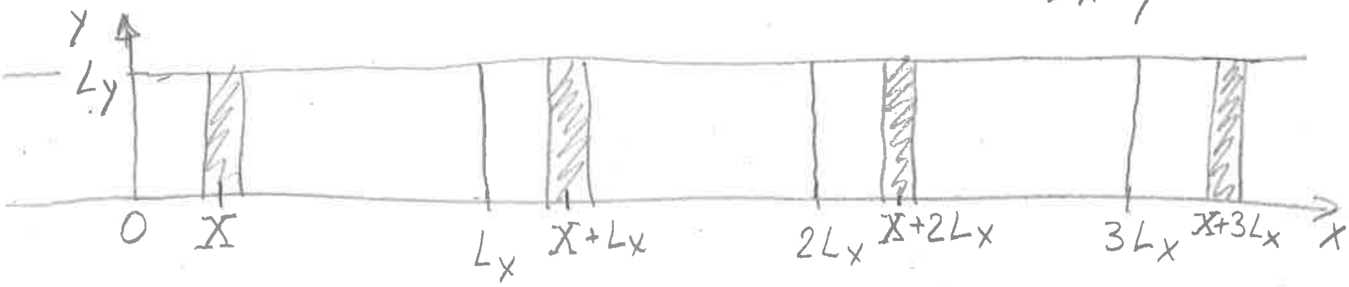
We thus get <u>m degenerate ground states</u>

$\Psi_0, \Psi_1, \dots, \Psi_{m-1}$; $\quad \Psi_j = t_x^j \Psi_0 \quad (j = 0 \dots m-1)$

which are eigenstates of t_y with different eigenvalues: $\quad t_y \Psi_j = e^{-j\cdot\frac{2\pi i}{m}} \Psi_j$

More detailed explanations to this calculation (see e.g. Appendix A of Bergholtz, Karlhede, PRB 77, 155308 (2008))

$$L_x L_y = 2\pi \ell_0^2 \cdot N_\phi$$



Landau-gauge vector potential $A_x = 0$, $A_y = Bx$.
Note that $\vec{A}$ is not periodic. Eigenfunctions of H:

$$\Psi_X(x,y) = \sum_{n=-\infty}^{\infty} e^{i\left(k_y - n\frac{L_x}{\ell_0^2}\right)y} \, e^{-(x-X-nL_x)^2/2\ell_0^2}$$

$$X = -k_y \ell_0^2, \quad k_y = -\frac{2\pi n_y}{L}, \quad n_y \in \mathbb{Z}$$

Magnetic translations:

$A_y = Bx$

$A_y(\vec{r}+\vec{a}) = B(x+a_x) = A_y(\vec{r}) + Ba_x$

$\vec{\nabla} f_{\vec{a}}(\vec{r}) = (0, Ba_x) \quad \Rightarrow \quad f_{\vec{a}}(\vec{r}) = Ba_x y$

$$\Rightarrow \varphi_{\vec{a}}(\vec{r}) = \frac{e}{\hbar c} B a_x y = \frac{a_x y}{\ell_0^2}$$

$$\tilde{T}_{\vec{a}} = e^{i\varphi_{\vec{a}}(\vec{r})} T_{\vec{a}}$$

$$\tilde{T}_{a_x \cdot \hat{x}} = e^{i a_x y/\ell_0^2} \cdot T_{a_x \cdot \hat{x}} = e^{i a_x y \ell_0^2} e^{a_x \partial_x}$$

$$= e^{a_x(\partial_x + iy/\ell_0^2)}$$

$$\tilde{T}_{a_y \cdot \hat{y}} = T_{a_y \cdot \hat{y}} = e^{a_y \partial_y}$$

(*) $\tilde{T}_{a_x \cdot \hat{x}}$ operator commutes with H if and only if it is consistent with periodic boundary conditions in y direction $\Leftrightarrow \quad a_x \cdot L_y/\ell_0^2 = 2\pi k, \quad k \in \mathbb{Z}$

$$\Leftrightarrow \quad a_x = k \cdot \frac{2\pi \ell_0^2}{L_y} = \frac{k L_x}{N_\Phi}, \quad k \in \mathbb{Z}$$

(*) $\tilde{T}_{a_y \cdot \hat{y}}$ operator commutes with H if and only if the eigenfunctions $\Psi_X(x,y)$ in the bottom of p. 10.21 are also eigenfunctions of $\tilde{T}_{a_y \cdot \hat{y}}$. This is the case if phases acquired by different terms in $\sum_n$ differ by integer multiple of 2π.

$$\longleftrightarrow \quad \frac{L_x}{\ell_0^2} a_y = 2\pi k, \quad k \in \mathbb{Z}$$

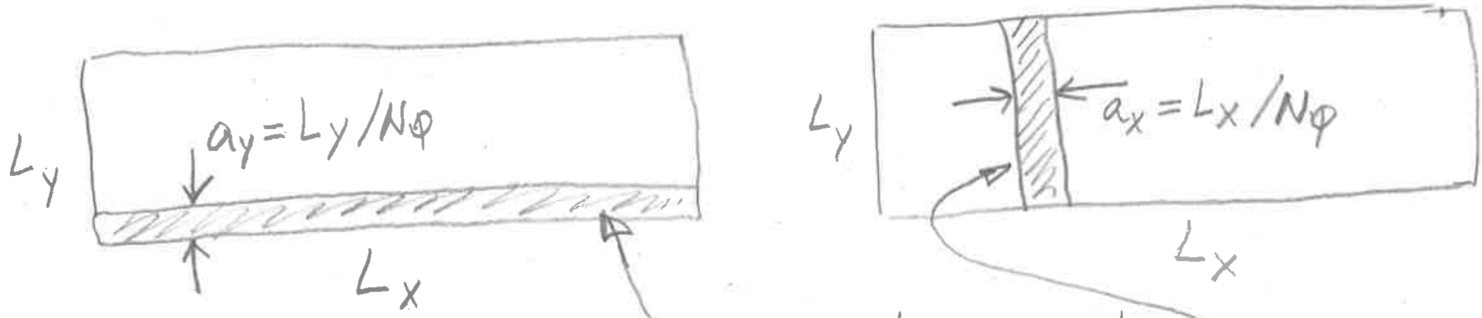
$$\longleftrightarrow \quad a_y = k \cdot \frac{2\pi \ell_0^2}{L_x} = \frac{k L_y}{N_\phi}, \quad k \in \mathbb{Z}$$

This yields the operators

$$t_x = \tilde{T}_{(L_x/N_\Phi)\cdot\hat{x}} \quad \text{and} \quad t_y = \tilde{T}_{(L_y/N_\Phi)\cdot\hat{y}}$$

that commute with H, as stated on page 10.20.

Physically, $a_x = L_x/N_\Phi$ and $a_y = L_y/N_\Phi$ is easy to understand:



The flux through this and through this area is exactly Φ_0.

The m-fold degeneracy of the $\nu = \frac{1}{m}$ ground state follows, see pp. 10.20 – 10.21. This argument is straightforwardly extended to $\nu = p/q$ ground state, leading to q-fold degeneracy.

(*) For a surface of genus g, the degeneracy is q^g for p/q FQHE state