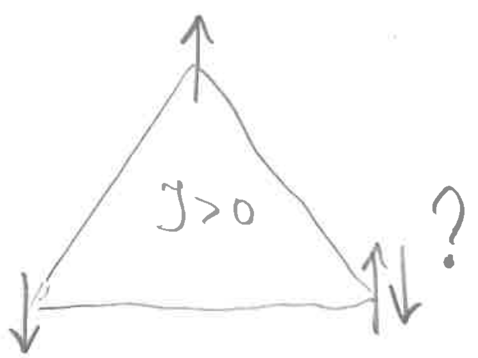
11. Spin liquids

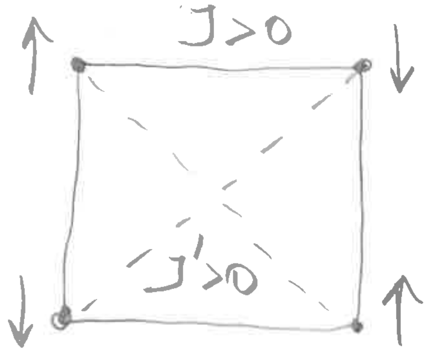
Review: L. Savary, L. Balents arXiv: 1601.03742

1D: Heisenberg antiferromagnet
ground state ≠ Néel state (↑↓↑↓↑..)
(quantum fluctuations)

2D, 3D bipartite lattices:
effect of quantum fluctuations is relatively weak
→ long-range order is expected to survive.

Non-bipartite lattices (triangular, kagome, ...):
spins are frustrated ⟹ it is not possible to minimize energies of all bonds even for classical spins;
bipartite lattices: frustration can be introduced by distant couplings





What is the quantum ground state?

Focus on spin-1/2 models

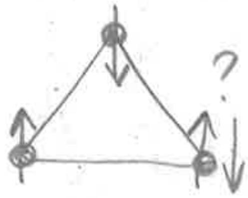
11.1 Spin liquids: RVB states

There is no commonly accepted definition of QSL...

AF on a triangular lattice: Resonating valence bond (RVB) state (Anderson) → all electrons form singlet bonds grouping is not unique → switching (resonating) between singlet configurations

Antiferromagnets of finite size, small spin

Geometric frustration

Homework: solve AF triangle 

Landau's principle of symmetry breaking

spin liquids → no SSB, no local order parameter

→ topological order

High-T_c superconductors

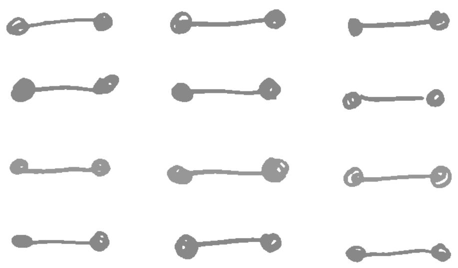
→ spin-liquid models, RVB

e.g., chiral spin liquid

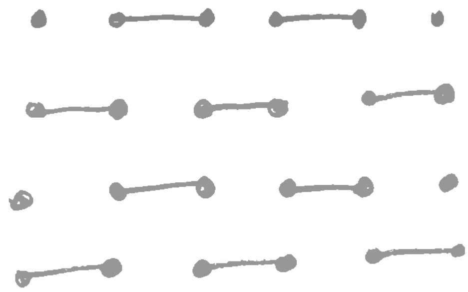
(originally: breaks TR and parity symmetries, but not SR symmetry; other kinds of chiral spin liquid with the same symmetry indistinguishable by Landau's SSB theory

beyond local order parameter: ground state degeneracy, topological entanglement entropy

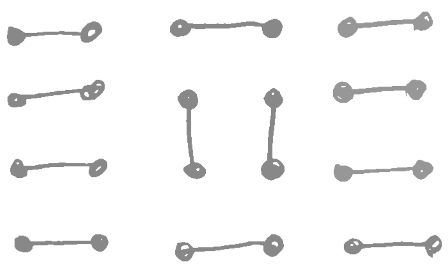
Valence-bond solid states



column VBS



staggered VBS



Valence-bond resonance
(columnar VBS)

not possible
in staggered VBS

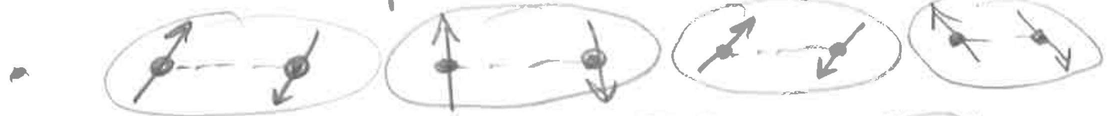
1D spin-$1/2$ chain: Majumbar-Ghosh model

J, J': $$H = \frac{J}{2} \sum_j \vec{S}_j \cdot \vec{S}_{j+1} + J' \sum_j \vec{S}_j \cdot \vec{S}_{j+2}$$

$J' = J/2$: $$H = \frac{J}{2} \sum_j \left(\vec{S}_j \cdot \vec{S}_{j+1} + \vec{S}_j \cdot \vec{S}_{j-1} + \vec{S}_{j-1} \cdot \vec{S}_{j+1} \right)$$

MG $$= \frac{J}{4} \sum_j \left(\vec{S}_{j-1} + \vec{S}_j + \vec{S}_{j+1} \right)^2 + \text{const}$$

Two exact ground states: VBS
pairs of spins forming singlets

I: 

II: 

translational symmetry broken!

2D: quantum fluctuation
lift degeneracy of VBS states

Consider two spins-$\frac{1}{2}$

$$H = J\,\vec{S}_1 \cdot \vec{S}_2$$

Néel state $|\uparrow\downarrow\rangle$ has energy $-J/4$
Exact ground state - singlet

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right), \quad \text{energy } -3J/4$$

$$E_{GS} < E_N$$

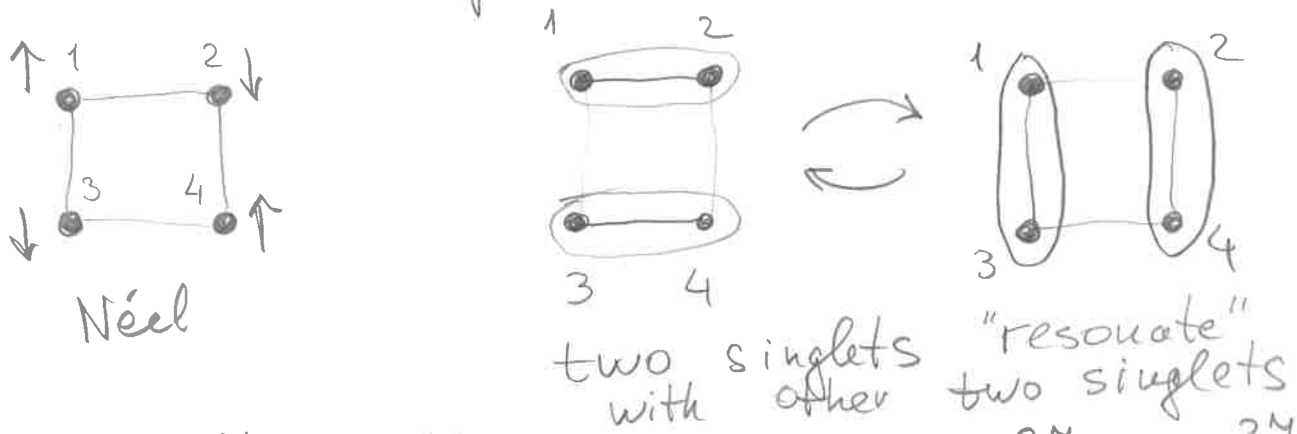
Néel state minimizes only $S_1^z S_2^z$;
Singlet minimizes all three terms (rotational invariance)

$\rightarrow$ $S_1^+ S_2^-$ destabilizes NS $(\uparrow\downarrow \rightarrow \downarrow\uparrow)$

Classically, spins in Néel state are antiparallel
Quantum-mechanically $\left([S^\mu, S^\nu] = i\hbar\epsilon^{\mu\nu\gamma} S^\gamma\right)$ it is only true for z-components; x and y components are totally uncertain

For the singlet state the two spins are always antiparallel, no matter the direction (individual spins are completely random) — <u>entanglement</u>
spins are more strongly correlated!

Consider a single square with four spins



$$E_N = -\frac{J}{4} \cdot 4 = -J$$

each bond $-\frac{J}{4}$ gain

$$E_{VBS} = -\frac{3J}{4} \cdot 2 = -\frac{3J}{2}$$

two bonds $-\frac{3J}{4}$ gain,
the other two 0.

Extending to 2D lattice $\rightarrow$
E_N is lower than E_{VBS} by $\frac{4}{3}$ factor.

Entanglement monogamy: the more entangled spin A is with spin B, the less it can be entangled with other spin C.

Perfect Bell's state (singlet) can occur only between A and one other spin B.

For large coordination number classical correlations (Néel) win over VBS.

But: 1. Multiple VBS state may resonate

2. Singlets can be formed between arbitrarily distant spins

$\Longrightarrow$ "entropic contribution"

Return to the single square.
Introduce notation $|(ij)\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle)$.
VBS states $|(12),(34)\rangle$ and $|(13),(24)\rangle$
are not eigenstates of the Hamiltonian:
H couples these two-singlet states

$$|\Psi_0\rangle \sim |(12),(34)\rangle + |(13),(24)\rangle$$

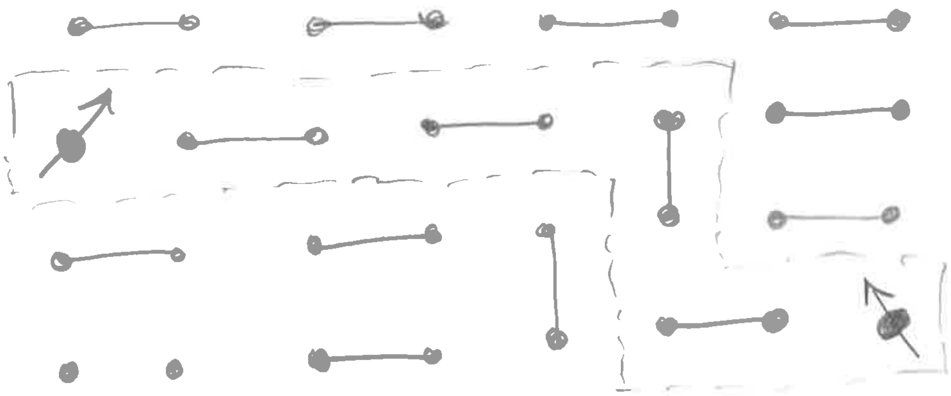
resonating valence-bond (RVB) state

In general, quantum spin liquids
have ground states of this type —
here, a coherent superposition of resonating
valence-bond states.

Return to VBS states

column VBS – easy to resonate
staggered VBS – difficult
$\Rightarrow$ expect lower energy for column VBS.

Excitations in the column VBS:

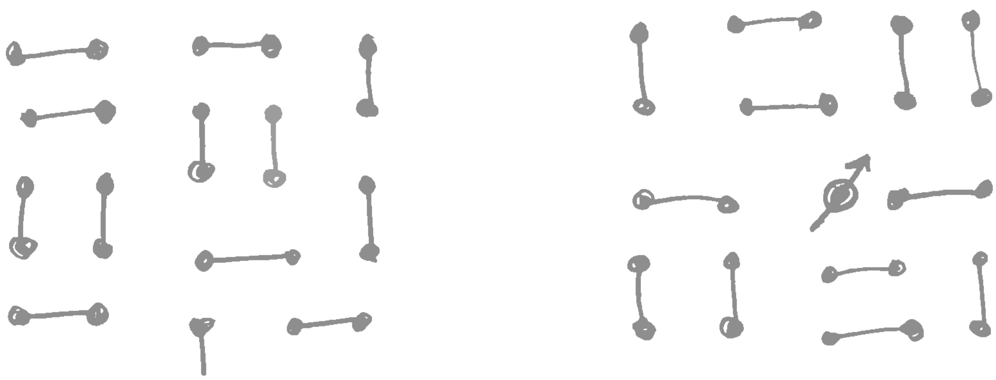


two spinons connected by a "staggered" string

Confinement of spinons (in contrast to 1D)

Valence-band solids:
Spinons are not elementary excitations; actual elementary excitations are gapped spin-1 magnons (triplet bonds)

Generic nearest-neighbor valence-bond state

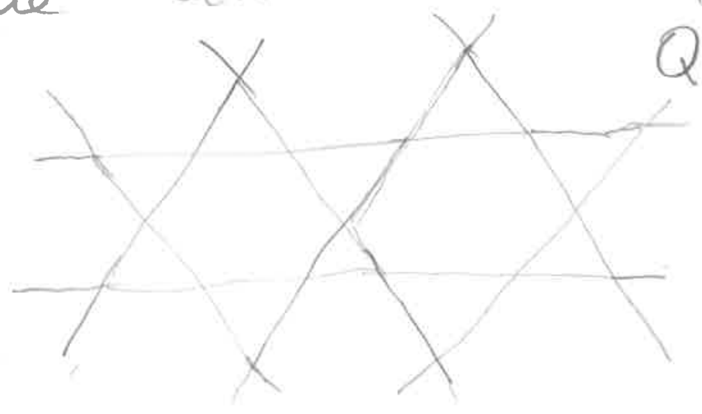


RVB: equal superposition of all such configurations
→ "liquid" state of valence bonds
RVB spin-liquid

a spinon in an RVB background, no disruption of the "solid" order
→ spinons are deconfined

Triangular lattice: coordination number $z=6$
spin order survives

kagome lattice: $z=4$
QSL - ?



Spin liquids are different from Néel state (NS breaks spin-rotation and translation symmetries) and VBS states (VBS respects spin-rotation symmetry, but breaks translation/rotation symmetry), since QSL states break none of the symmetries of the Hamiltonian.

How to distinguish between spin liquids and other states?
Entanglement (topological order)!

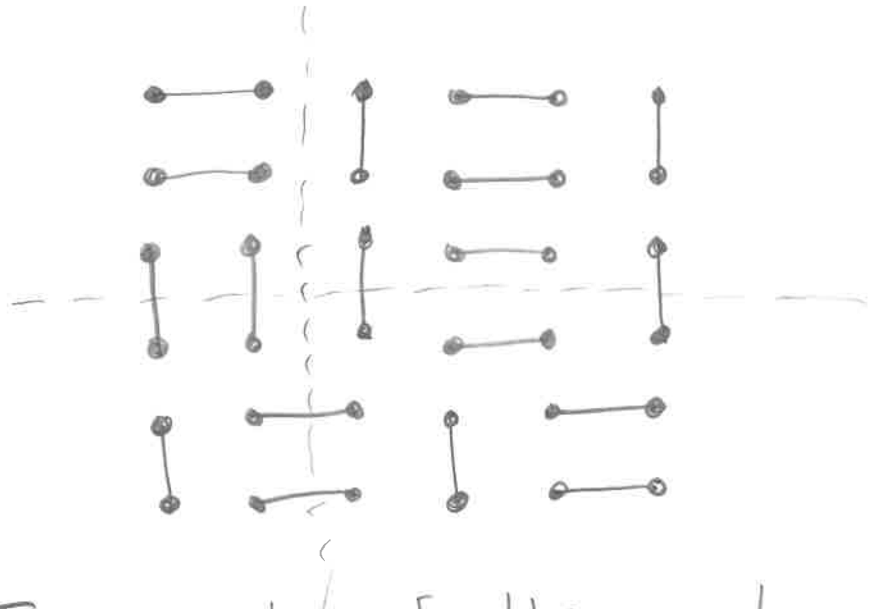
Néel: product state, no entanglement
VBS: "almost product" state (only neighboring spins are correlated)
QSL: huge superposition of valence-bond configurations, cannot be reduced to an "almost product" state by local unitary operations (mutually commuting)

Zero-T state: quantum fluctuations are crucial in QSL → interference, Berry phase → topology.
Ground-state degeneracy (as in FQHE).

11.2 $\mathbb{Z}_2$ topological order in RVB spin liquids.

$L_x \times L_y$ rectangle with periodic boundary conditions
$\rightarrow$ torus

Assume L_x and L_y to be both even for simplicity



6x6 VB configuration

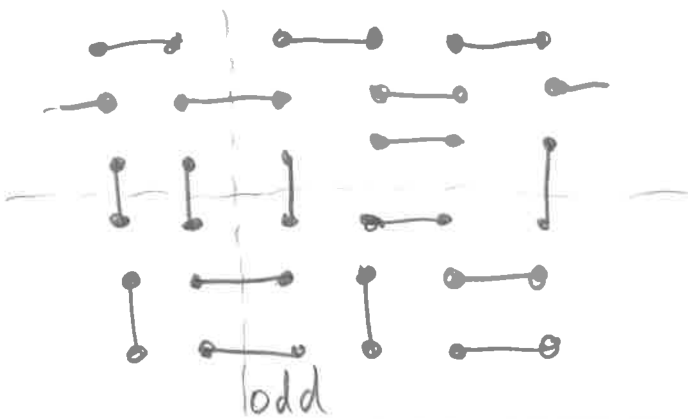
here, any horizontal or vertical cut always intersects an even number of bonds

The parity of the number of intersections is the same for any x-cut and any y-cut in a given configuration

Ground-state degeneracy is described by two $\mathbb{Z}_2$ quantum numbers
($\mathbb{Z}_2$ spin liquid)

$\mathbb{Z}_2 \times \mathbb{Z}_2$: four distinct sectors $(D=4)$

even-even, even-odd, odd-even, odd-odd



RVB state preserves this classification!

$D=4$, $\underset{\text{genus}}{g=1}$, degeneracy D^g

11.3 Quantum dimer models

1988: Rokhsar - Kivelson
quantum dimer model (QDM)

→ non-magnetic state for undoped Mott insulator

Dimer: SU(2) singlet for neighboring spins (no long-range dimers)

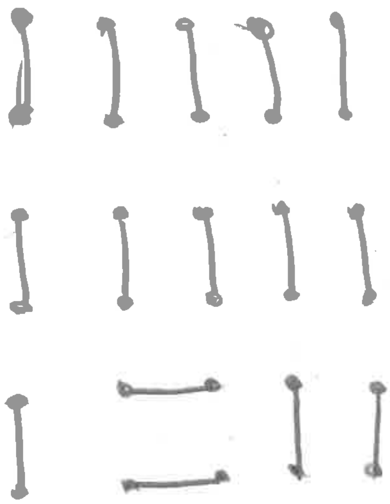
Original Hilbert space is truncated (only valence bonds, no overlaps)

Dimer covering: complete orthonormal set of basis states:

$$\langle D | D' \rangle = \begin{cases} 0, & \text{if } D \neq D' \\ 1, & \text{if } D = D' \end{cases}$$

For singlet coverings overlap is never zero

|a)



|b>

overlap $\langle a|b\rangle$

Square lattice:

$$H_{RK} = \sum_{\substack{\text{4-site}\\ \text{plaquettes}}} \Big[-t\big(|\,\shortparallel\,\rangle\langle\,=\,| + h.c. \big) + V\big(|\,\shortparallel\,\rangle\langle\,\shortparallel\,| + |\,=\,\rangle\langle\,=\,| \big) \Big]$$

Terms with

t - flip } parallel dimers
V - count }

otherwise, the state gets annihilated

Other lattices: plaquettes or resonance loops

$$H^{v=t} = v \sum_{\square} \big(|\,\shortparallel\,\rangle - |\,=\,\rangle \big)\big(\langle\,\shortparallel\,| - \langle\,=\,| \big); \quad |RK\rangle = \frac{1}{\sqrt{N_C}} \sum_C |C\rangle$$

all dimer coverings

Triangular lattice (square lattice + diagonals)

$$H_{RK} = \sum \Big[-t\big(|\,/\!/\,\rangle\langle\,=\,| + |\,=\,\rangle\langle\,/\!/\,| \big) + V\big(|\,/\!/\,\rangle\langle\,/\!/\,| + |\,=\,\rangle\langle\,=\,| \big) \Big]$$

When $t = V$

projectors $P = |\Psi\rangle\langle\Psi|$

$$H_{RK} = \sum_{\diamond} 2t\, |\Psi\rangle\langle\Psi|$$

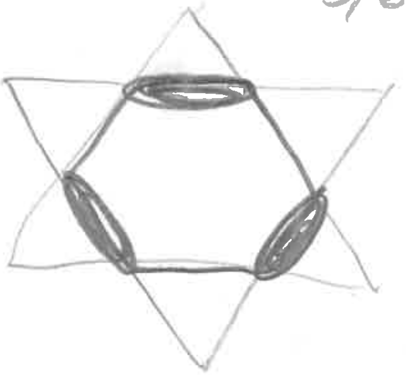
$$|\Psi(\diamond)\rangle = \frac{1}{\sqrt{2}}\big(|\,/\!/\,\rangle - |\,=\,\rangle \big)$$

Kagome lattice:

flippable plaquettes of various lengths

12-site star:

3/6



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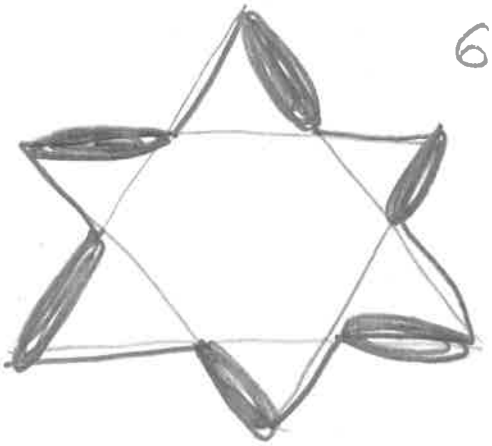

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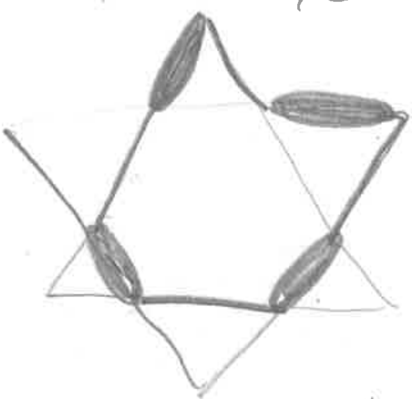

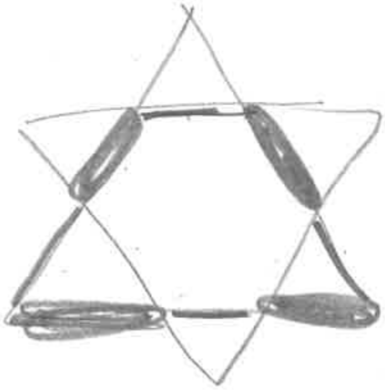

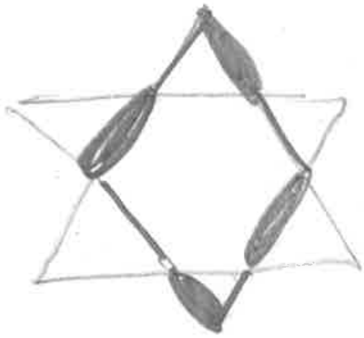

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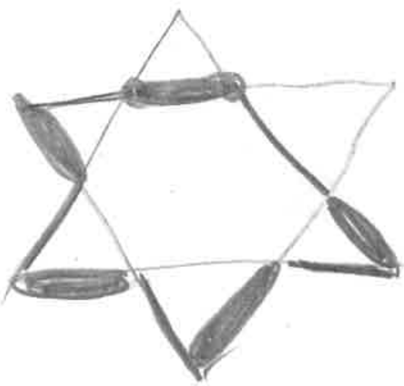
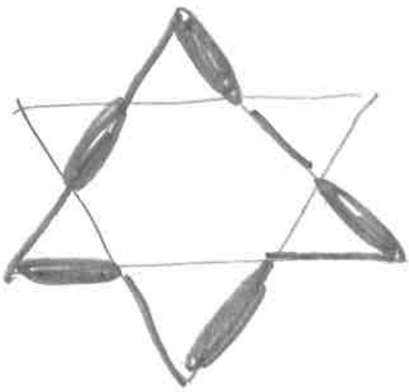




all configurations are flippable → v-term is trivial

Rokhsar - Kivelson point

$t = V$

$$|\Psi_{GS}\rangle = \frac{1}{\sqrt{N_C}} \sum_C |C\rangle$$

equal superposition of all dimer tilings (coverings).

Flippable plaquettes: penalty v
benefit t

Nonflippable: zero energy

$\Rightarrow \quad E_{GS} \geq \min(0, N_{plaq}(V-t))$

Periodic boundary conditions:
ground state degeneracy according to the number of topological sectors

Square lattice: RK point - liquid GS, critical point separating valence bond solids with broken translational symmetry (U(1) spin liquid)

Triangular lattice: RK point is inside
Kagome → a stable spin-liquid phase (Z_2 universality class)

$\mathbb{Z}_2$ RVB liquid for QDM

Periodic boundary conditions:
ground-state space – 4 degenerate gapped states;
no symmetry is broken,
all correlations decay exponentially

Numerics: $0.8 < v/t < 1$
triangular

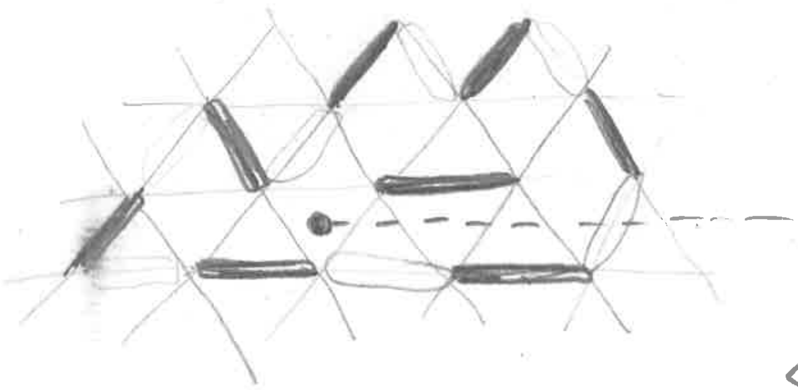
Generic feature of non-bipartite lattices

Excitations: monomers
(deconfined pairs)

visons:

$$|\Psi_{\text{vison}}\rangle = \sum_{C} (-1)^{n_D(C)} |C\rangle$$

where $n_D(C)$ is the number of dimers crossing the dashed line associated with the vison ("dislocation")



visons $\bullet$ live on dual lattice

$$\langle \Psi_{\text{vison}} | \Psi_{GS} \rangle = 0$$

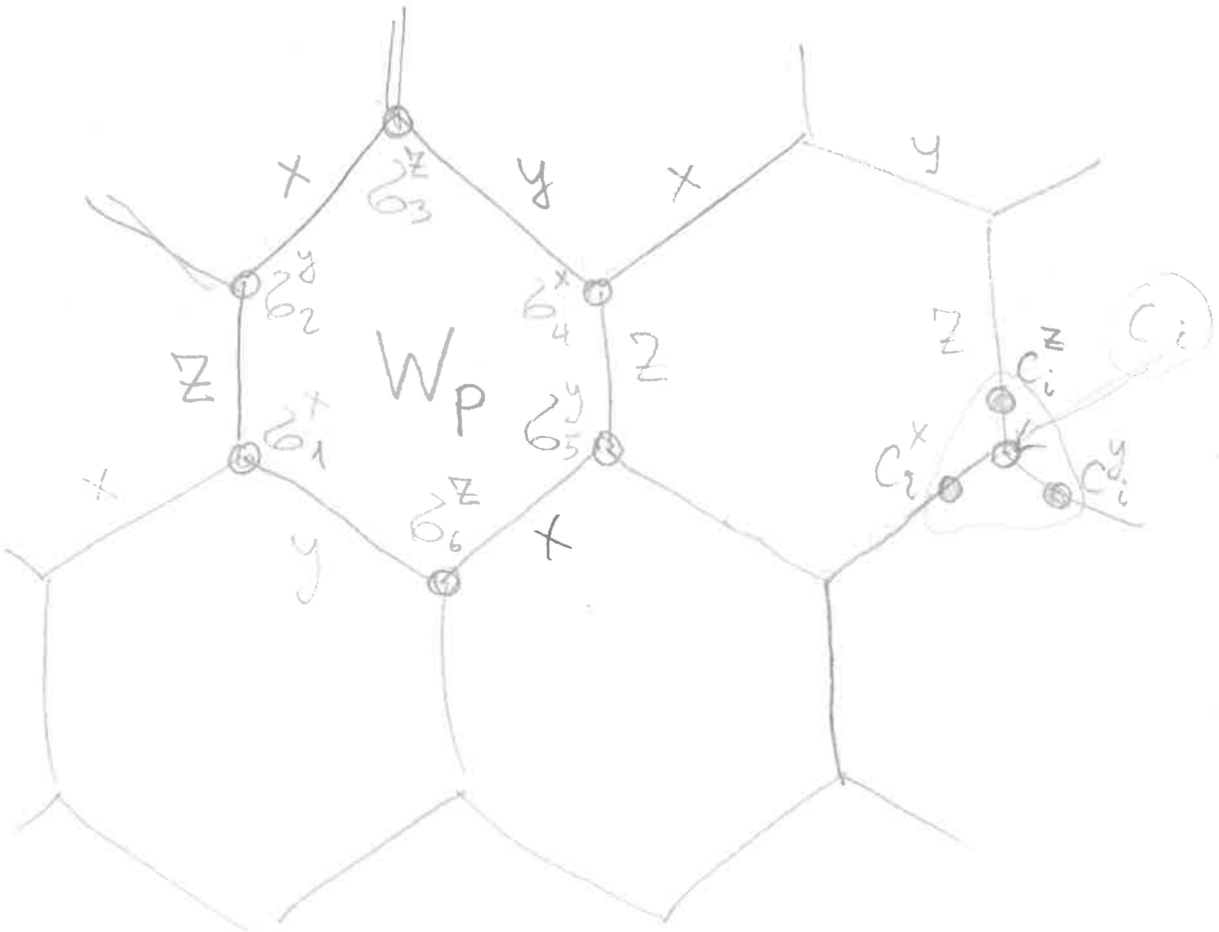
11.4. Kitaev honeycomb model

– exactly soluble QSL model

see A. Kitaev. arXiv: cond-mat/0506438

Spin-$\frac{1}{2}$ with anisotropic exchange on a honeycomb lattice

$$H = J_x \sum_{\langle ij\rangle \in x} \sigma_i^x \sigma_j^x + J_y \sum_{\langle ij\rangle \in y} \sigma_i^y \sigma_j^y + J_z \sum_{\langle ij\rangle \in z} \sigma_i^z \sigma_j^z$$



Plaquette operator

$$W_p = \sigma_1^x \sigma_2^y \sigma_3^z \sigma_4^x \sigma_5^y \sigma_6^z \,; \qquad [W_p, H] = 0$$

$W_p = \pm 1$ is conserved

local $\mathbb{Z}_2$ symmetry of H:

states with different eigenvalues of W_p are allowed (W_p is measurable)

"Majorana partons"

define $\sigma^\mu = i\, c\, c^\mu$

Majorana fermion operators, $c^\dagger = c$

$$H = \frac{i}{4} \sum_{\langle ij \rangle} \hat{A}_{ij}\, c_i c_j$$

$$\hat{A}_{ij} = 2 J_{\gamma_{ij}}\, \hat{u}_{ij}$$

$$\hat{u}_{ij} = i\, c_i^{\gamma_{ij}} c_j^{\gamma_{ij}}, \qquad \gamma_{ij} = \mu \text{ if } \langle ij \rangle \in \mu$$

→ exact transformation if we keep only states such that

$$D_i |\Psi\rangle = |\Psi\rangle, \qquad D_i = c_i^x c_i^y c_i^z c_i$$

(act on arbitrary state with projector $P = \prod_i \frac{1+D_i}{2}$ ⟹ produce such states)

11.17

Spin-fermion transformation

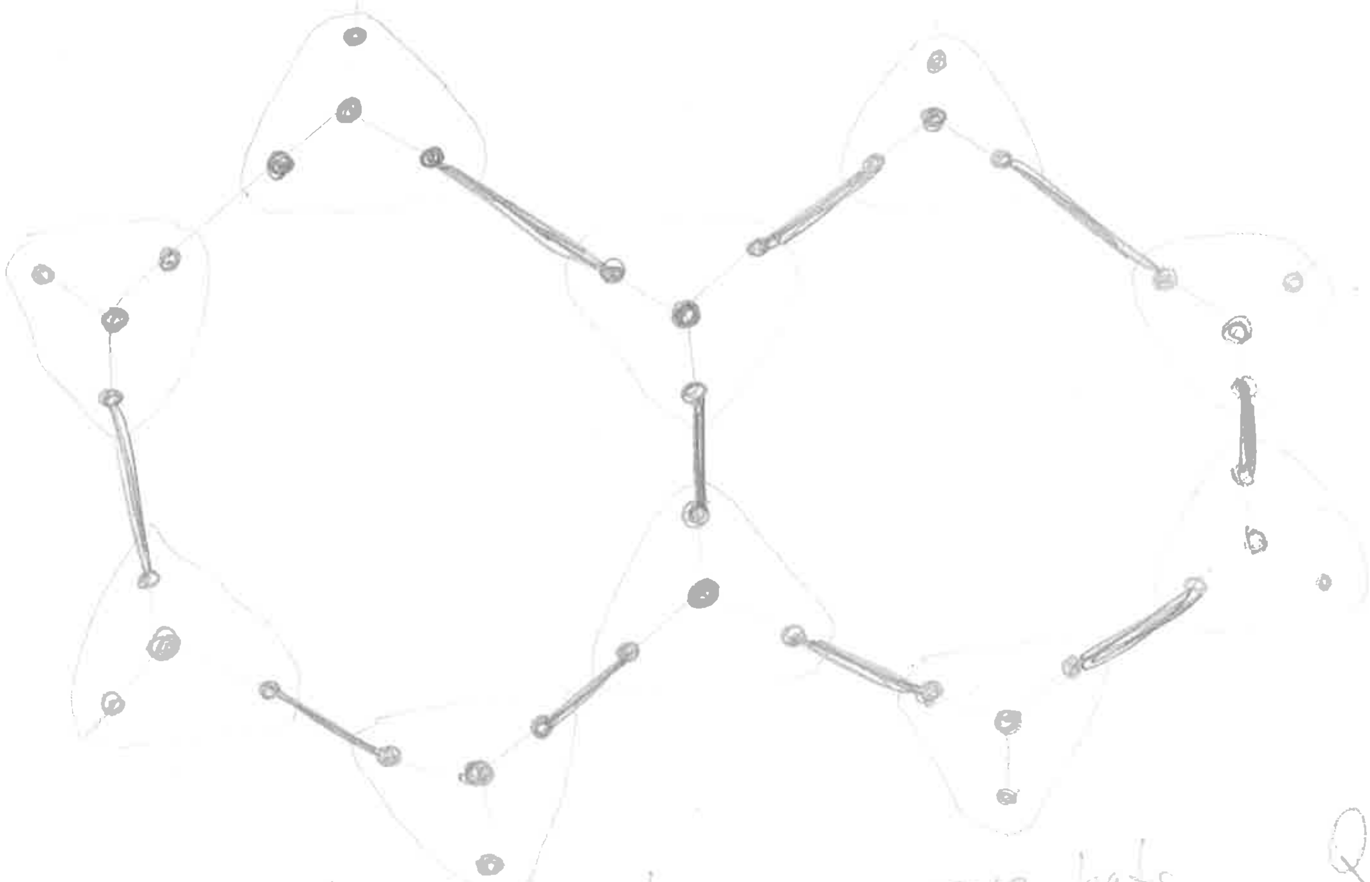
n fermionic modes a_k, a_k^+ $(k=1..n)$

Linear combinations

$$c_{2k-1} = a_k + a_k^+, \quad c_{2k} = \frac{a_k - a_k^+}{i}$$

– Majorana operators $(j = 1, \dots, 2n)$

$$c_j^2 = 1, \quad c_j c_\ell = - c_\ell c_j \quad \text{if } j \neq \ell$$



In each sector remove hats from $\hat{A}$: $\quad Q = e^A$

$$H(A) = \frac{i}{4} \sum_{jk} A_{jk} c_j c_k \qquad e^{-iH(A)} c_k e^{iH(A)} = \sum_j Q_{jk} c_j$$

A real skew-symmetric matrix

Since in H each c_i^α appears only once, $[\hat{u}_{ij}, \hat{u}_{kl}] = 0$

$\Rightarrow$ $[\hat{u}_{ij}, H] = 0$

work in each eigenvalue sectors where $u_{ij} = \pm 1$ $\rightarrow$ free fermions

$$W_p = \prod_{\langle ij \rangle \in p} \hat{u}_{ij}$$

Ground state $W_p = 1 \quad \forall p$

Satisfied by $u_{ij} = \begin{cases} 1 & \text{sublattice A} \\ -1, & \text{sublattice B} \end{cases}$

Dispersion for c fermions:

$$\epsilon_k = \pm 2 \left| J_x e^{i \vec{k} \cdot \vec{n}_1} + J_y e^{i \vec{k} \cdot \vec{n}_2} + J_z \right|$$

$$\vec{n}_{1,2} = \left(\pm \sqrt{3}/2, \, 3/2 \right)$$

for $|J_x| \leq |J_y| + |J_z|$ and $|J_y| \leq |J_z| + |J_x|$ and $|J_z| \leq |J_x| + |J_y|$

$\rightarrow$ two zero-energy Dirac points

two phases: gapless and gapped

Excitations: "defect" plaquettes with $W_p = -1$

To move the defect to the neighboring plaquette
— change the sign of one link $\langle ij \rangle$ shared by two plaquettes

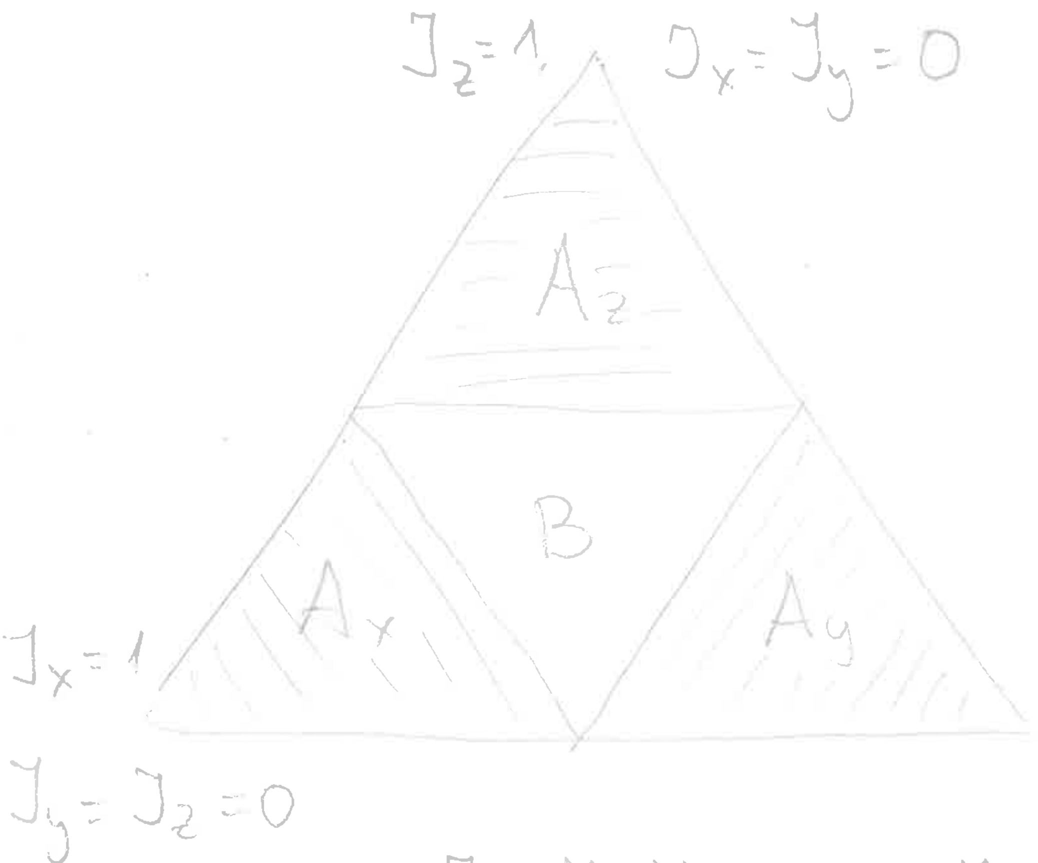
z-link $\hat{u}_{ij} = i c_i^z c_j^z$

anticommutes with $\sigma_i^z = i c_i c_i^z$

To move around the central plaquette apply W_p

For $W_p = -1 \rightarrow$ phase shift π

Anyons (two swaps are not equivalent to unity)



A_x, A_y, A_z gapped

B - gapless

$J_x, J_y, J_z \geq 0$, $J_x + J_y + J_z = 1$

Perturbation theory

Consider phase A_z

near the corner: $|J_x|, |J_y| \ll |J_z|$

$$H = H_0 + V$$

$$H_0 = -J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z$$

$$V = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y$$

assume $J_z > 0$

$J_x = J_y = 0$: highly degenerate GS

$\uparrow\uparrow$ or $\downarrow\downarrow$ along z links,

no common direction, $E_0 = -N J_z$

effective spins

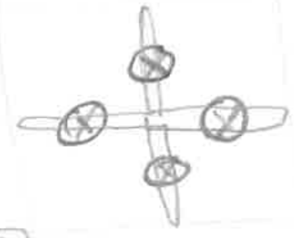
$$H_{eff} = -N \frac{J_x^2 + J_y^2}{4 J_z} - \frac{J_x^2 J_y^2}{16 |J_z|^3} \sum_p Q_p$$

$Q_p = \sigma^y_{left} \sigma^y_{right} \sigma^z_{up} \sigma^z_{down}$

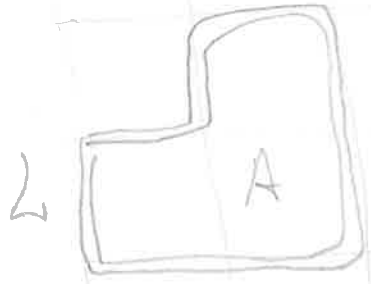
11.21

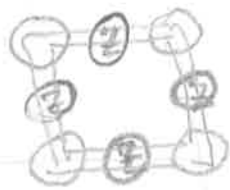
11.5 Toric code on the square lattice.

Spins-$\frac{1}{2}$ on the links



$$S_s = \prod_{i \in s} \sigma_i^x \quad \text{star}$$





$$P_p = \prod_{i \in p} \sigma_i^z \quad \text{plaquette}$$

loop operator on the contour L

S_s and P_p Hermitian, square to 1 $\to \pm 1$ eigenvalues

$$H_{tc} = -K_p \sum_p P_p - K_s' \sum_s S_s$$

$$[S_s, S_{s'}] = 0; \quad [P_p, P_{p'}] = 0$$

$$[S_s, P_p] = 0 \quad \forall p, s$$

Overlap at even number of edges

Ground state: $S_s = P_p = +1$ for all p, s

Excitations: stars or plaquettes -1

GS (and any state specified by $\{S_s, P_p\}$) is 4-fold degenerate. $\prod_s S_s = +1$ since every σ_i^x appears twice

Similarly, $\prod_p P_p = +1$

N sites — N−1 independent choices of star eigenvalues and N−1 choices of plaquette eigenvalues

$\rightarrow 2^{(N-1)+(N-1)}$ choices of $\{S_s, P_p\}$

But in total 2^{2N} spin states

$\rightarrow$ 4 independent states for each $\{S_s, P_p\}$

In any local basis the state is highly entangled

Consider σ_i^x basis

$S_s = 1$: even number of $\sigma_i^x = -1$

color these links $\rightarrow$ closed loops

Since P_p is offdiagonal in this basis

$\rightarrow$ superposition of loop states

PBC (torus) 4 inequivalent choices

Base states with fixed parity of the number of loops in x and y directions (product over all $\sigma_i^x = S_i$ on a column of horizontal bonds)

A state with $P_p = +1 \quad \forall p$:

take as a "base" state a product state (eigenstate of σ_i^x)

Act with projectors

$$Q_p = \frac{1 + P_p}{2} \qquad (P_p \to +1)$$

$$|0\rangle = \prod_p Q_p \left(\otimes_i |\sigma_i^x = S_i\rangle \right)$$

↑ chosen to satisfy $S_s = +1$

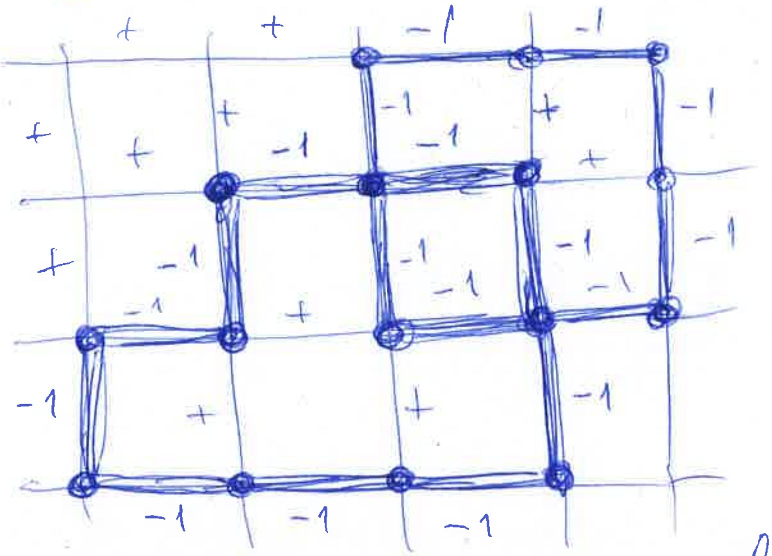
For example

take $S_i = 1$ everywhere, act with $\prod_p Q_p$

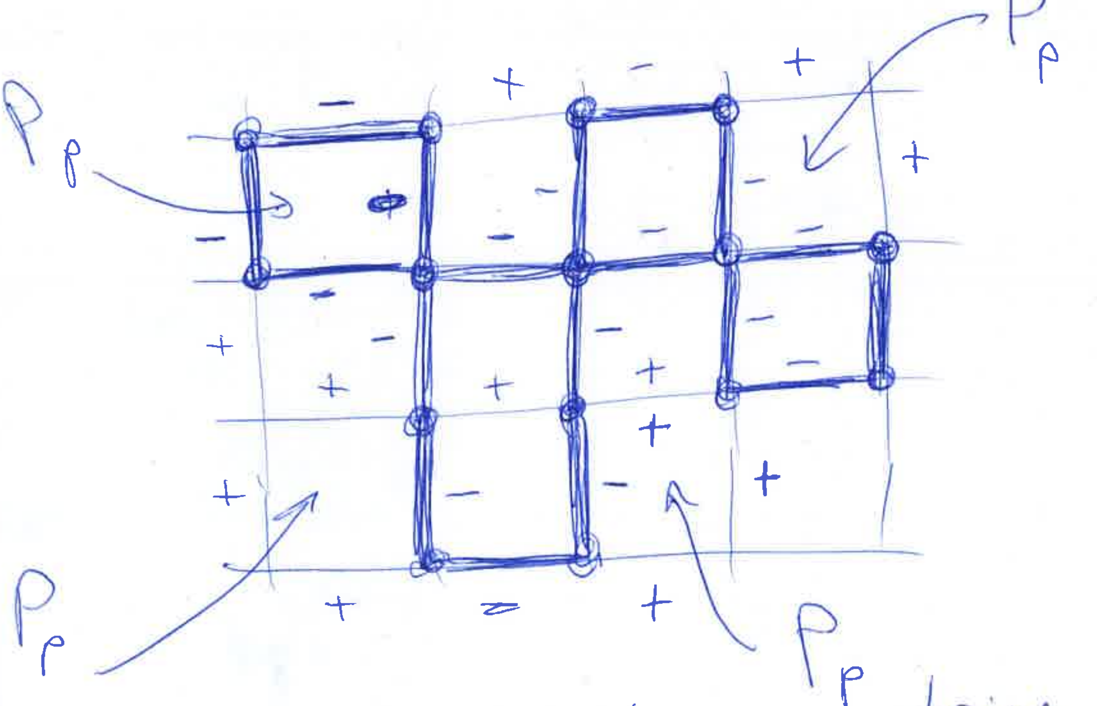
→ superposition of loop states

Using $Q_p = Q_p P_p$, we see that states in which S_i are related by flipping spins around a minimal square plaquette are the same after projection

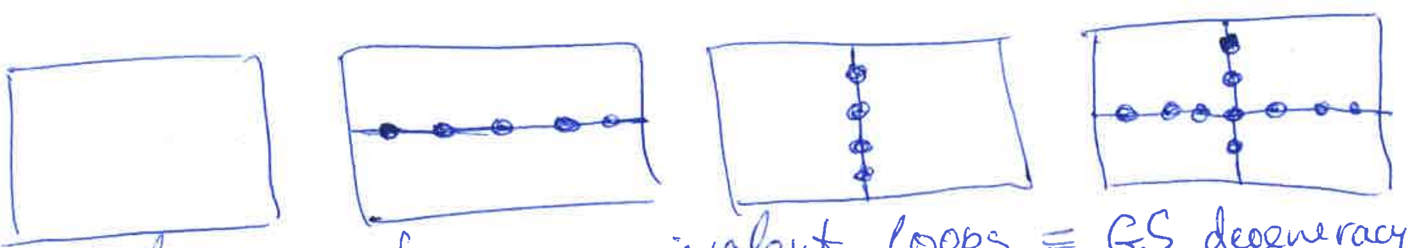
Starting with loop states in σ^x basis



Apply P_p : add new loops or deform existing loops



Cannot break a string but can eliminate loops that do not wind around the torus



4 classes of nonequivalent loops = GS degeneracy
Parity of winding loops

Parity is unchanged by P_p

$\Rightarrow$ states of different parity are orthogonal

"Topological degeneracy"

Different degenerate states cannot be distinguished locally

Local operator $\hat{O}$ (product of σ_i^r over finite set $< N$)

$$\langle m|\hat{O}|n\rangle = \bar{O}\,\delta_{nm}$$

$\rightarrow$ "topological protection"

Excitations: anyons

choose a single star or plaquette to be negative: $S_{s_0} = -1$ or $P_{p_0} = -1$

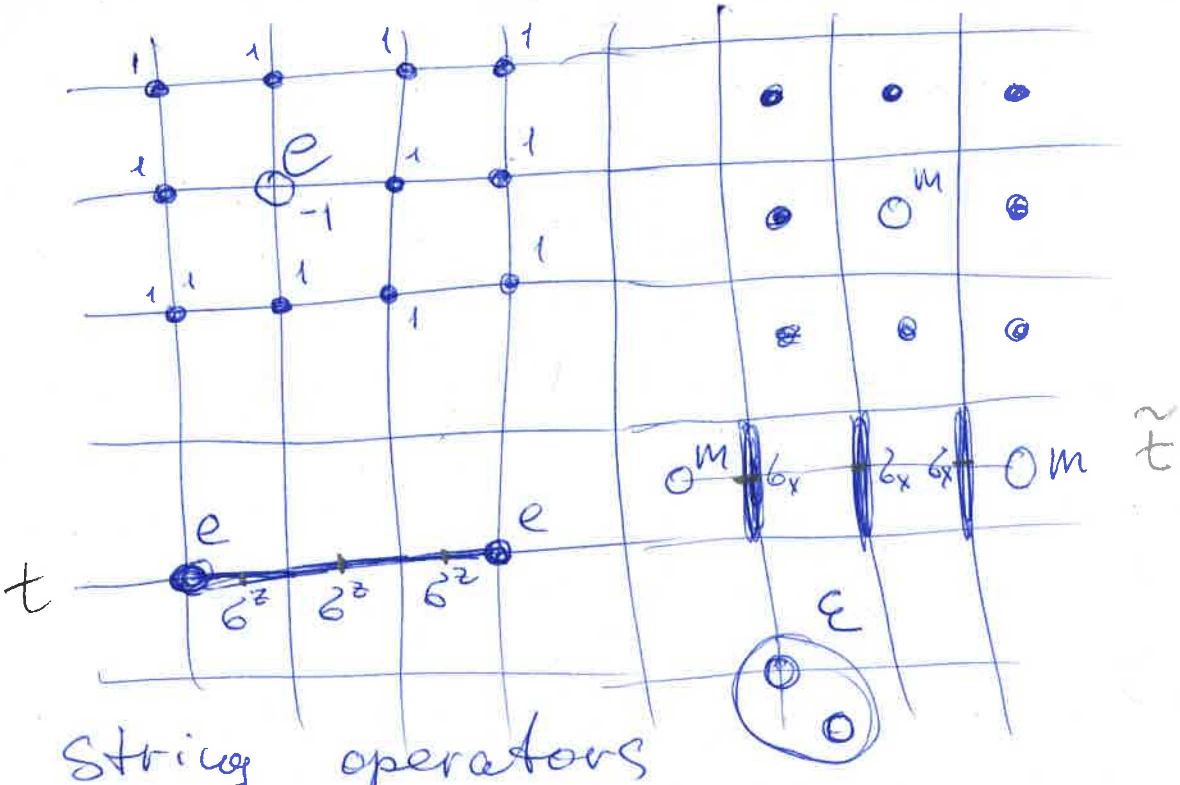
Excitation energy $2K_s'$ or $2K_p$

"electric" (e) or "magnetic" (m) particles

cannot be created individually (each spin is shared by two stars)

single σ_i^z creates 2 e

string of σ_i^z creates 2 e at the ends



String operators

$$S^x(t) = \prod_{j \in \text{string } t} \sigma_j^x , \qquad S^z(\tilde{t}) = \prod_{j \in \text{string } \tilde{t}} \sigma_j^z$$

$$[S^x(t_1), S^x(t_2)] = 0 \qquad [S^z(\tilde{t}_1), S^z(\tilde{t}_2)] = 0$$

bosons   bosons

if we move e particle (applying σ_i^z) around m (or vice versa) the state acquires -1 factor.

mutual statistics "semionic" anyonic

$$|\Psi_{fin}\rangle = \prod_{i \in \lambda} \sigma_i^z |\Psi_{init}\rangle$$

"Stokes' theorem"

$$\prod_{i \in \lambda} \sigma_i^z = \prod_{p \in A} P_p , \quad \lambda = \partial A \rightarrow$$

one m particle inside A (one $P_p = -1$)

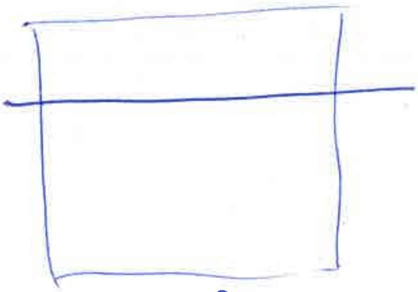
$$\Rightarrow |\Psi_{fin}\rangle = -|\Psi_{init}\rangle$$

Entanglement

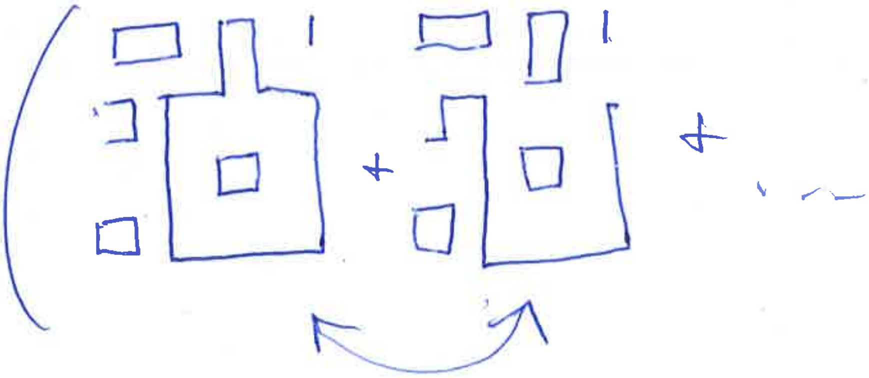
two anyons feel each other
even when they are arbitrarily far apart
→ structure in the wavefunction

String operator creates
two particles at energy cost $4K_s$
for arbitrary length of string

Ground state is a massive superposition
string reshuffles



$\prod \sigma_i^z$



Measure: Entanglement entropy

$$\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

$$S(A) = -\mathrm{Tr}_A \, \rho_A \log \rho_A$$

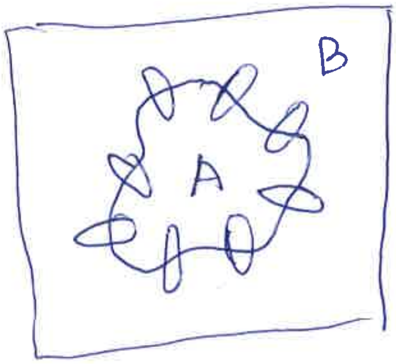
$S(A) = 0$ for $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

Typical ground state (2D)

$$S(A) \sim S_0 L - \gamma$$ (Area law)

length of the boundary between A and B

Area-law: $L = |\partial A|$



in 3D - surface area

S_0 - non-universal

Excitation gap:
γ is universal for smooth boundary

→ topological entanglement entropy

toric code: $\gamma = \ln 2$

negative correction:
it is impossible to deform the ground state to obtain $S_0 = 0$
since $S(A) \geq 0$ by definition

⇒ $\gamma \neq 0$ direct indication of long-range (topological) entanglement

Finite temperature: volume law
mutual information:
$$I(A,B) = S(A) + S(B) - S(A \cup B)$$
area law (but involves classical contributions)