# 5. Topology in one dimension
(SSH model, Kitaev chain, AKLT...)

$\Rightarrow$ introduce the concept of SPT (symmetry-protected topological) phases

$\Rightarrow$ topological end modes (edge states)

$\Rightarrow$ Majorana fermions

## 5.1. Discrete symmetries

1D gapped systems:
topological invariants require the presence of unbroken symmetries
$\rightarrow$ symmetry-protected topological phases
(cf. Verresen et al., PRB (2017))

Hamiltonian written in real space:

$H_{r'\alpha', r\alpha}$ - matrix elements between states $|r\alpha\rangle$

other degree of freedom (e.g., spin)

Translationally invariant
$\Rightarrow$ Fourier transformation

$$\frac{1}{N}\sum_{r',r} H_{r'\alpha', r\alpha}\, e^{i\vec{k}(\vec{r}-\vec{r}\,')} = H_{\alpha',\alpha}(\vec{k})$$

Transpose:

$$\frac{1}{N}\sum_{r'r} H^T_{r'\alpha', r\alpha}\, e^{i\vec{k}(\vec{r}'-\vec{r})} = H_{\alpha,\alpha'}(-\vec{k})$$

$$H \to H(\vec{k}),\quad H^T \to H^T(-\vec{k}),\quad H^* \to H^*(-\vec{k})$$

$$H^\dagger \to H^\dagger(\vec{k})$$

Use Pauli matrices $\tau$ for sublattices, $\sigma$ for physical spin

- Time-reversal symmetry $\mathcal{T}$

$$i\hbar\partial_t\Psi = H\Psi \Leftrightarrow -i\hbar\partial_t\Psi^* = H^*\Psi^*$$

$U_T$ unitary matrix to be found $\Leftrightarrow -i\hbar\,\partial_t(U_T\Psi^*) = U_T H^* U_T^\dagger (U_T\Psi^*)$

Time reversal: $U_T\Psi^*$ satisfies Schrödinger equation with $t\to -t$

$$\Psi \to U_T\Psi^*,\quad H \to U_T H^* U_T^\dagger$$

$t\to -t$: $-i\hbar\,\partial_t(T\Psi^*) = H(T\Psi^*)$

$$\Rightarrow U_T H^* U_T^\dagger = H,\quad U_T H^*(-k) U_T^\dagger = H(k)$$

real space Fourier

Example:

$$H(\vec{k}) = (\cos k_x a + \cos k_y a)\tau_x + \sin k_z a\,\tau_z$$

$U_T = \tau_x$: $\tau_x H^*(-\vec{k})\tau_x = H(\vec{k})$

$$\Psi(k) \to \tau_x\Psi^*(-\vec{k})$$

TRS with spin:

$\mathcal{T} = i\sigma_y K$, where $K$ denotes complex conjugation

Spin $\frac{1}{2}$: $\mathcal{T}^2 = -1$; $\psi$ and $\mathcal{T}\psi$ - Kramers pair

$\mathcal{T}$ is an anti-unitary operator

$$\left[ \mathcal{T}\psi \cdot \mathcal{T}\varphi = (\psi \cdot \varphi)^* \right]$$

$$\boxed{\text{TRS:} \quad [H, \mathcal{T}] = 0, \quad \mathcal{T} = U_T K \\ U_T^+ H^*(k) U_T = H(-k)}$$

- Space inversion (parity)

$P\psi(-\vec{r})$ satisfies the same Schrödinger equation as $\psi(\vec{r})$:

$$H\psi(-\vec{r}) = i\hbar \partial_t \psi(-\vec{r})$$

$$PHP^+P\psi(-\vec{r}) = i\hbar \partial_t P\psi(-r)$$

$$\Rightarrow \boxed{PH(-\vec{r}, -\vec{r}\,')P^+ = H(\vec{r}, \vec{r}\,') \\ PH(-\vec{k})P^+ = H(\vec{k})}$$

Example: two Dirac nodes:
$$H(\vec{K} + \vec{q}) = q_x \tau_x + q_y \tau_y$$
$$H(-\vec{K} + \vec{q}) = -q_x \tau_x + q_y \tau_y$$

$P = \tau_x$; adding $m\tau_z$ breaks parity

- Particle-hole (charge-conjugation) symmetry

$U_C \psi^*$ satisfies SE for $\psi$

$\{H, C\} = 0$ with $C = U_C K$

$$\boxed{\begin{aligned} C: \quad & H \to U_C H^* U_C^\dagger \\ & U_C^\dagger H^*(\vec{k}) U_C = -H(-k) \end{aligned}}$$

Graphene: $U_C = \tau_z$
broken by $m\tau_z$

Charge conjugation for spin 1/2 particles appears in the context of superconductivity (Bogolinbov-de Gennes matrices)

- Chiral (or sublattice) symmetry

$S: \quad \{H, U_S\} = 0$ ($U_S$ unitary)

$$\boxed{U_S^\dagger H(\vec{k}) U_S = -H(\vec{k})}$$

Graphene: $U_S = \tau_z$
Dirac mass breaks chirality

Chiral symmetry implies that the Hamiltonian can be brought to off-diagonal form:

$$H = \begin{pmatrix} 0 & h \\ h^+ & 0 \end{pmatrix}$$

Chiral symmetry must be present whenever H enjoys T and C:

$$H \xrightarrow{T} \mathcal{U}_T H^* \mathcal{U}_T^+ = H \xrightarrow{C} \mathcal{U}_C H^* \mathcal{U}_C^+ = -H$$

$$S = CT, \quad \mathcal{U}_S = \mathcal{U}_C \mathcal{U}_T$$

Inverse is not true: chiral symmetry can exist when both C and T are broken.

- PT symmetry

  may exist when neither P nor T are separately present.

Composition of P and T

$$H(\vec{k}) \to P U_T H^*(\vec{k}) U_T^+ P^+$$

$$\boxed{PT: \quad H(\vec{k}) = \mathcal{U}_{PT} H^*(k) \mathcal{U}_{PT}^+}$$

Both $(PT)^2 = 1$ and $(PT)^2 = -1$ are possible

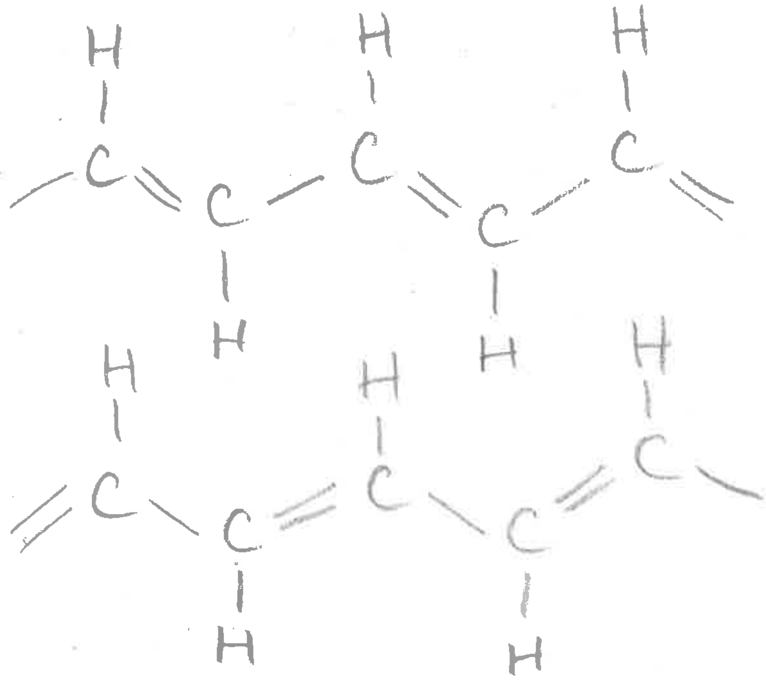
$$\hat{H} = h_x \tau_x + h_y \tau_y + h_z \tau_z$$

$h_z = 0$: $\mathcal{U}_{PT} = \tau_x$; $h_x = 0$: $\mathcal{U}_{PT} = \tau_z$; $h_y = 0$, $\mathcal{U}_{PT} = 1$

# 5.2 Staggered 1D lattice

Su-Schrieffer-Heeger (SSH) model
→ Shockley model

SSH model of Polyacetilene



trans-$(CH)_x$

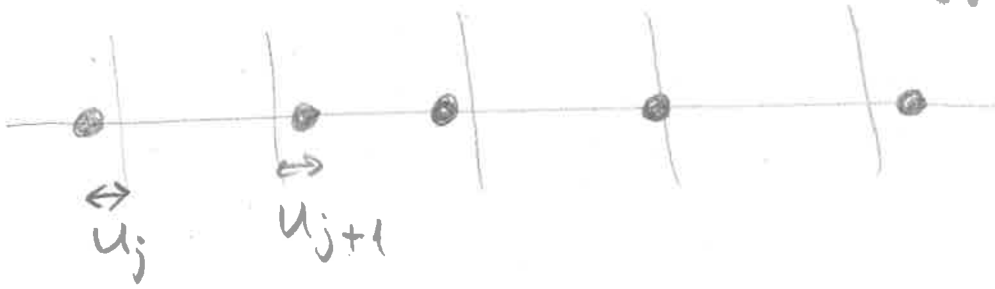
Experimental puzzle:
charge carriers carry charge only, no spin

Conductivity is uncorrelated with magnetic susceptibility

SSH: charge carriers are charged solitons that carry no spin

$$H = \sum_j \left[ -t(u_{j+1}-u_j)\,|j\rangle\langle j+1| + \text{H.c.} \right] + \sum_j \frac{\lambda}{2}\left(u_{j+1}-u_j\right)^2$$

lattice elastic energy




$$t(u_{j+1}-u_j) \simeq t - \frac{\alpha}{2}\left(u_{j+1}-u_j\right)$$

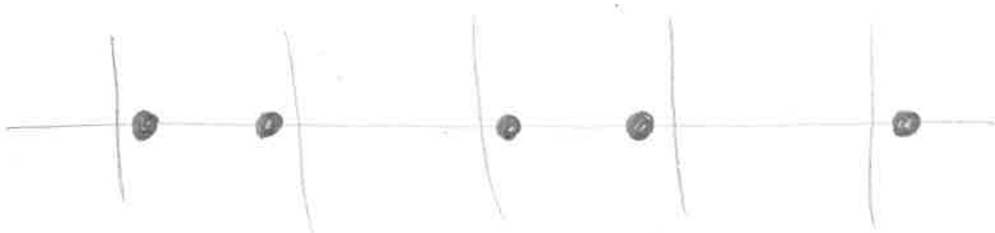
Peierls instability

spontaneous dimerization, $\quad u_j = u(-1)^j$, $\delta t = \alpha u$

$$\Delta E + \Delta U \approx -\frac{(\alpha u)^2 L}{\pi v_F} \ln\left(\frac{v_F}{\alpha |u|}\right) + 2\lambda N u^2$$

electronic $\quad$ elastic $\quad$ logarithmic divergence

$\longrightarrow$



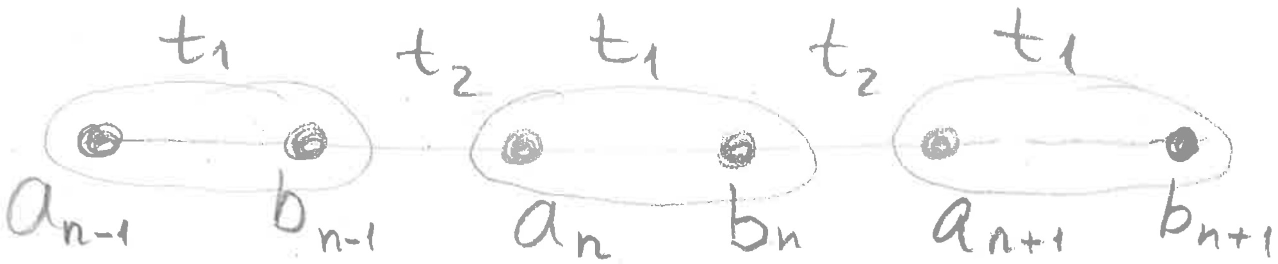
Odd number of sites $\rightarrow$ frustration

Even number $\rightarrow$ pairs of defects $\quad \leftarrow$ topological defect (excitation)

Staggered (dimerized) lattice:

$$H = \left(t_1 \sum_{n=N_0}^{N} c^+_{n,b} c_{n,a} + H.c.\right) \quad \text{hopping within unit cell}$$

$$+ \left(t_2 \sum_{n=N_0}^{N-1} c^+_{n+1,a} c_{n,b} + H.c.\right) \quad \text{between unit cells}$$




$$\langle a,n|\hat{H}|b,n\rangle = t_1$$
$$\langle b,n|\hat{H}|a,n+1\rangle = t_2$$

$$\Phi_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \qquad \begin{aligned} E a_n &= t_1 b_n + t_2^* b_{n-1} \\ E b_n &= t_2 a_{n+1} + t_1^* a_n \end{aligned}$$

Consider a semi-infinite chain, $n \geq 0$

Solution with $E=0$

$$a_n = \left(-\frac{t_1^*}{t_2}\right)^n a_0, \qquad b_n = 0$$

can exist if $|t_2| > |t_1|$

$\Rightarrow$ the chain starts with the weak link. The state is exponentially localized at the edge, lives exclusively on sublattice $a$

If chain terminates at $n=L$:

$n \leq L$: $\quad a_n = 0, \quad b_n = \left(-\frac{t_1}{t_2^*}\right)^{L-n} b_L$

The $E=0$ state lives on sublattice b

Consider infinite system ($N_0 \to -\infty$, $N \to \infty$)

Fourier transformation:

1BZ $\quad H(k) = \vec{d}(k) \cdot \vec{\tau} = \begin{pmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{pmatrix}$

$k \in \left[-\frac{\pi}{a}, \frac{\pi}{a}\right]$

$$= \begin{pmatrix} 0 & t_1 + t_2^* e^{-ika} \\ t_1^* + t_2 e^{ika} & 0 \end{pmatrix}$$

$$\vec{d}(k) = \left(|t_1|\cos\theta_1 + |t_2|\cos(ka+\theta_2), -|t_1|\sin\theta_1 - |t_2|\sin(ka+\theta_2), 0\right)^T$$

restricted to XY plane

Real hopping amplitudes ($\theta_1, \theta_2 = 0$)

$\Rightarrow$ Rice – Mele model with $\Delta = 0$ (equal on-site energies)

Eigenenergies:

$$E(k) = \pm \left| t_1 + t_2^* e^{ika} \right|$$

Consider real hopping amplitudes
Rice – Mele with $\Delta = 0$

$$E(k) = \pm\sqrt{4t^2\cos^2\frac{ka}{2} + 4\delta^2\sin^2\frac{ka}{2}}$$

$$= \pm\sqrt{t_1^2 + t_2^2 + 2t_1t_2\cos(ka)} = \pm|\vec{d}(k)|$$

Band gap $E_g = 2|t_1 - t_2|$
closes when $t_1 = t_2$

Symmetries:

$$H^*(k) = H(-k)$$

TRS: $\mathcal{U}_T = \mathbb{1}$, $\mathcal{T} = K$

PHS: $\mathcal{U}_C = \tau_z$, $C = \tau_z K$

Chiral: $\mathcal{U}_S = \tau_z$

$(\mathcal{U}_S = \mathcal{U}_C \mathcal{U}_T)$

Chiral symmetry of a 2×2 Hamiltonian $H(k) = \vec{d}(k)\cdot\vec{\tau}$ ensures that there is a plane in which $\vec{d}(\vec{k})$ lies for all $\vec{k}$; special point $|d(k)| = 0$
$\Rightarrow \mathbb{R}^2\setminus\{0\}$ (hence mapping $S^1 \to S^1$ in 1D)

Proof:

Any unitary $\mathcal{U}$ can be written as $U = u_0 + i\vec{u}\cdot\vec{\sigma}$ (up to irrelevant phase factor)

with real $u_0, u_1, u_2, u_3$.

Let us first use this with complex coefficients

$U^\dagger = u_0^* - i\vec{u}^*\cdot\vec{\sigma}$ and $U^{-1} = (u_0 - i\vec{u}\cdot\vec{\sigma})/\det U$

$|\det U| = 1 \Rightarrow \det U = e^{i\varphi}$

$\Rightarrow u_0^* = e^{-i\varphi} u_0, \quad (i\vec{u})^* = -e^{-i\varphi}(i\vec{u})$

Multiply $U$ by $e^{-i\varphi}$, such that $u_0$ real

$\Rightarrow$ all coefficients are real

Now let $\mathcal{U}_S = u_0 + i\vec{u}\cdot\vec{\tau}$

be unitary that transforms

$H(k) = \vec{d}(k)\cdot\vec{\tau}$ to $-\vec{d}(k)\cdot\vec{\tau}$

for all $k$:

$$\mathcal{U}_S H \mathcal{U}_S^\dagger = (u_0 + i\vec{u}\cdot\vec{\tau})(\vec{d}\cdot\vec{\tau})(u_0 - i\vec{u}\cdot\vec{\tau})$$

$$= u_0^2(\vec{d}\cdot\vec{\tau}) + iu_0[(\vec{u}\cdot\vec{\tau})(\vec{d}\cdot\vec{\tau}) - (\vec{d}\cdot\vec{\tau})(\vec{u}\cdot\vec{\tau})] + (\vec{u}\cdot\vec{\tau})(\vec{d}\cdot\vec{\tau})(\vec{u}\cdot\vec{\tau})$$

$$= u_0^2\vec{d}\cdot\vec{\tau} - 2u_0[\vec{u}\times\vec{d}]\cdot\vec{\tau} + 2((\vec{u}\cdot\vec{d})\vec{u})\cdot\vec{\tau} - \vec{u}^2\,\vec{d}\cdot\vec{\tau}$$

$$\Rightarrow \quad U_S H U_S^\dagger = \vec{\tilde{d}} \cdot \vec{\tau},$$ 

where

$$\vec{\tilde{d}} = (u_0^2 - \vec{u}^2)\cdot\vec{d} + 2u_0[\vec{u}\times\vec{d}] + 2(\vec{u}\cdot\vec{d})\,\vec{u}$$

Chiral symmetry: $\vec{\tilde{d}}(k) = -\vec{d}(k)$ for all $k$

$[\vec{u}\times\vec{d}]$ is orthogonal to $\vec{d}$

$\Rightarrow u_0 = 0$ (if $\vec{u} \parallel \vec{d}$, then $\vec{\tilde{d}}(k) = \vec{d}(k)$ – does not work)

$$\Rightarrow \vec{\tilde{d}} = -\vec{u}^2\vec{d} + 2(\vec{u}\cdot\vec{d})\,\vec{u}$$

should be 1 should be 0

$\vec{u}\cdot\vec{d}(k) = 0$ for all $k$

$\Rightarrow \vec{d}(k)$ belongs to the plane orthogonal to $\vec{u}$ as $k$ varies

SSH model: $U_S = \tau_z$

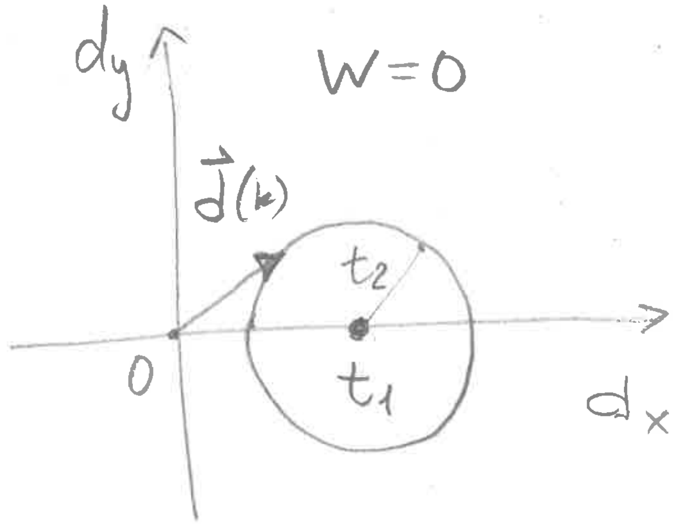
$$\Rightarrow d_z(k) \equiv 0$$

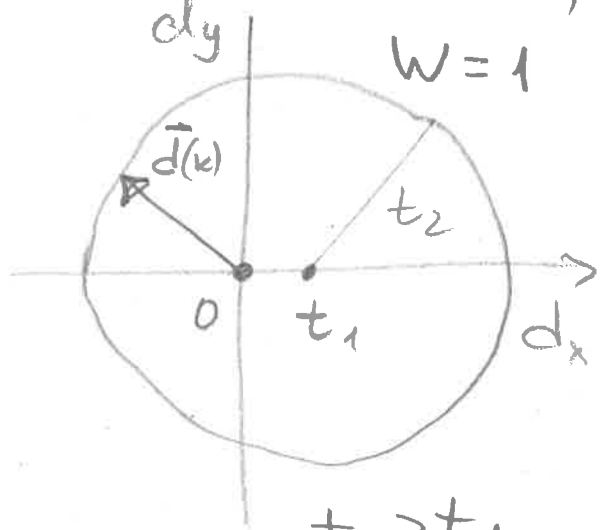
Let $\varphi(k)$ be a polar angle of $\vec{d}(k)$ in XY plane where $\vec{d}(k)$ lives

Winding number around $|\vec{d}|=0$ (no gap)

$$W = \frac{1}{2\pi}\oint_{1BZ} dk \,\frac{\partial\varphi(k)}{\partial k} = \frac{\varphi(\frac{\pi}{a}) - \varphi(-\frac{\pi}{a})}{2\pi}$$

Another useful form: $W = \frac{1}{2\pi}\int_{1BZ}\left[\vec{e}_d(k)\times\frac{d}{dk}\vec{e}_d(k)\right]dk$,
where $\vec{e}_d = \vec{d}/|\vec{d}|$.




$t_1 > t_2$ "trivial phase"

$t_2 > t_1$ "topological phase"

for the given (natural) choice of the unit cell (exercise)

$t_2 = 0$


$t_1 = 0$


end states

Left ⇔ Right: $t_1 \leftrightarrow t_2$, $W_{left} = 1 - W_{right}$

More generally:

Phase of $H_{12}(k)$

$$W = \frac{1}{2\pi i} \int_{1BZ} dk \, \frac{d \log\left(t_1 + t_2^* e^{-ika}\right)}{dk}$$

Chiral symmetry $\rightarrow$ $\vec{d}(k)$ in the plane

$\rightarrow$ winding number is defined

Symmetry-protected topological (SPT) phase:

Chiral symmetry guarantees topology and guarantees existence of $E=0$ end state

SSH model: example of 1D topological insulator.

Zak phase

Chiral symmetry:

$$H(k) = \begin{pmatrix} 0 & H_{12} \\ H_{12}^* & 0 \end{pmatrix}$$

Let $\varphi(k)$ be the phase of $H_{12}(k)$

Lowest-energy wave function

$$\psi(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^{i\varphi(k)} \\ 1 \end{pmatrix}$$

Berry connection

$$A(k) = i\langle \psi(k) | \partial_k \psi(k) \rangle = -\frac{1}{2}\frac{\partial \varphi(k)}{\partial k}$$

Berry phase

$$\gamma = \int_{-\pi/a}^{\pi/a} dk\, A(k) = -\pi W$$

is quantized in the presence of chiral symmetry ($W$ integer)

Berry phase for 1D systems $\equiv$ Zak phase

in general, non-gauge-invariant, but the change of the Zak phase signifies transition between topo phases.

Number of edge states is determined by $W$

$$H(k) = \begin{pmatrix} 0 & t_1 + t_2 e^{-2ika} \\ (t_1 + t_2 e^{-2ika})^* & 0 \end{pmatrix} \Rightarrow W = 2$$

$$E a_n = t_1 b_n + t_2^* b_{n-2}$$
$$E b_n = t_2 a_{n+2} + t_1^* a_n$$

$\rightarrow$ two decoupled chains

Number of edge states is topological invariant

# Higher-dimensional Hamiltonians

Chirality:

$$H(k) = \begin{pmatrix} 0 & \hat{q}(k) \\ \hat{q}^+(k) & 0 \end{pmatrix}, \quad \hat{q} \text{ - unitary matrix}$$

$W$ is the winding number of $\det \hat{q}$

$$\det \hat{q} = e^{i\varphi}$$

unitary matrix

$$W = \frac{1}{2\pi} \int d\varphi = \frac{1}{2i\pi} \int_{1BZ} \frac{d(\log \det \hat{q})}{dk} dk$$

$\hat{q} = e^{i\hat{a}}$, $\hat{a}$ - Hermitian, eigenvalues $\lambda_j$

$$\det \hat{q} = e^{i \operatorname{Tr}(\hat{a})}, \qquad \varphi = \operatorname{Tr} \hat{a}$$

$$\operatorname{Tr}\{q^+ dq\} = \operatorname{Tr}\{e^{-i\hat{a}} d(e^{i\hat{a}})\}$$

$$= \sum_j e^{-i\lambda_j} d(e^{i\lambda_j}) = \sum_j i\, d\lambda_j = i\, d\varphi$$

Hellmann-Feynman theorem

Hence, $$W = W[\hat{q}] = \frac{1}{2\pi i} \int_{1BZ} \operatorname{Tr}\left[\hat{q}^+ \frac{d\hat{q}}{dk}\right] dk$$

General argument relating symmetries and topology (in particular, the existence of edge states) in 1D SPT phases: "symmetry fractionalization".

# 5.3 One-dimensional Kitaev model (Majorana chain)

$$H_K = \sum_{j=1}^{\bar{N}} \Big[ -\underbrace{t}_{\text{hopping}} \left( c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) + \underbrace{\Delta}_{\text{pairing amplitude}} \left( c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger \right) \Big] - \sum_{j=1}^{N} \underbrace{\mu}_{\text{chemical potential}} \left( c_j^\dagger c_j - \frac{1}{2} \right)$$

$\bar{N} = N$ for PBC
$\bar{N} = N-1$ for open BC

1D p-wave superconductor of spinless fermions

fluctuations in 1D $\Rightarrow$ no true long-range order, but the model can be justified by proximity effect (a semiconducting wire in proximity to a bulk superconductor + spin-orbit interaction + magnetic field).

The model can be solved by the Bogoliubov transformation (exercise)

$$H_k(k) = \begin{pmatrix} \xi_k & \Delta_k \\ \Delta_k^* & -\xi_k \end{pmatrix} \text{ in the Nambu basis of } \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$$= \vec{d}(k) \cdot \vec{\sigma} = \begin{pmatrix} 0 \\ -\Delta_k \\ \xi_k \end{pmatrix} \cdot \vec{\sigma},$$

where $\xi_k = -t\cos(ka) - \mu$, $\Delta_k = i\Delta \sin(ka)$

Winding number $W$ of $\vec{d}(k)$ in the YZ plane: bulk band topology (exercise)

# Representation of the Kitaev model by Majorana fermions.

The operators of spinless fermions can be written in terms of Hermitian operators, Majoranas, as

$$c_j = \frac{1}{2}\left(\gamma_{1,j} + i\gamma_{2,j}\right),$$
$$c_j^\dagger = \frac{1}{2}\left(\gamma_{1,j} - i\gamma_{2,j}\right),$$

$$\{\gamma_m, \gamma_n\} = 2\delta_{nm} \quad \text{(Clifford algebra)}$$

Majoranas ("real fermions" as opposed to conventional "complex fermions"): creation and annihilation operators are equal, $\hat{\gamma} = \hat{\gamma}^\dagger$; $\gamma^2 = 1$.

In terms of these operators, the Hamiltonian of the Kitaev model takes the form:

$$H_K = \frac{i}{2}\sum_j \left[(-t+\Delta)\gamma_{1,j}\gamma_{2,j+1} + (t+\Delta)\gamma_{2,j}\gamma_{1,j+1}\right]$$
$$-\frac{i}{2}\mu\sum_j \gamma_{1,j}\gamma_{2,j}$$

Take $\mu = 0$ and $t = \Delta$ (for $t = -\Delta$: different pairing of Majoranas):

$$H_K = it\sum_j \gamma_{2,j}\gamma_{1,j+1}$$

Open boundary condition: operators $\gamma_{1,1}$ and $\gamma_{N,2}$ are absent

$\rightarrow$ zero modes

Define non-local fermion operators:

$$d_j = \frac{1}{2}\left(\gamma_{2,j} + i\gamma_{1,j+1}\right)$$
$$d_j^{\dagger} = \frac{1}{2}\left(\gamma_{2,j} - i\gamma_{1,j+1}\right)$$

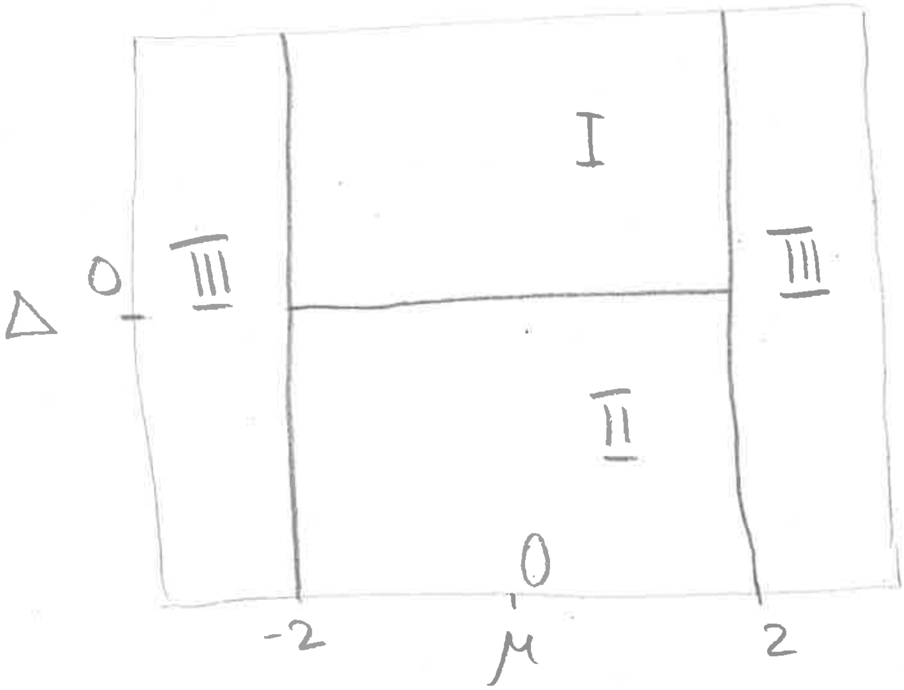
$$d_j^{\dagger} = \frac{1}{2}\left[-i(c_j - c_j^{\dagger}) - i(c_{j+1} + c_{j+1}^{\dagger})\right].$$

$$2d_j^{\dagger}d_j - 1 = i\gamma_{2,j}\gamma_{1,j+1}$$
$$= 1 + (c_j^{\dagger} - c_j)(c_{j+1}^{\dagger} + c_{j+1})$$

With $d_N \equiv \frac{1}{2}\left(\gamma_{N,2} + i\gamma_{1,1}\right)$,

$$H_K = t\sum_{j=1}^{N-1}\left(2d_j^{\dagger}d_j - 1\right) + \underbrace{\epsilon_N}_{=0}\left(2d_N^{\dagger}d_N - 1\right)$$

The $d_N$-state may be occupied or empty, $d_N^{\dagger}d_N = 1, 0$, with no energy cost.

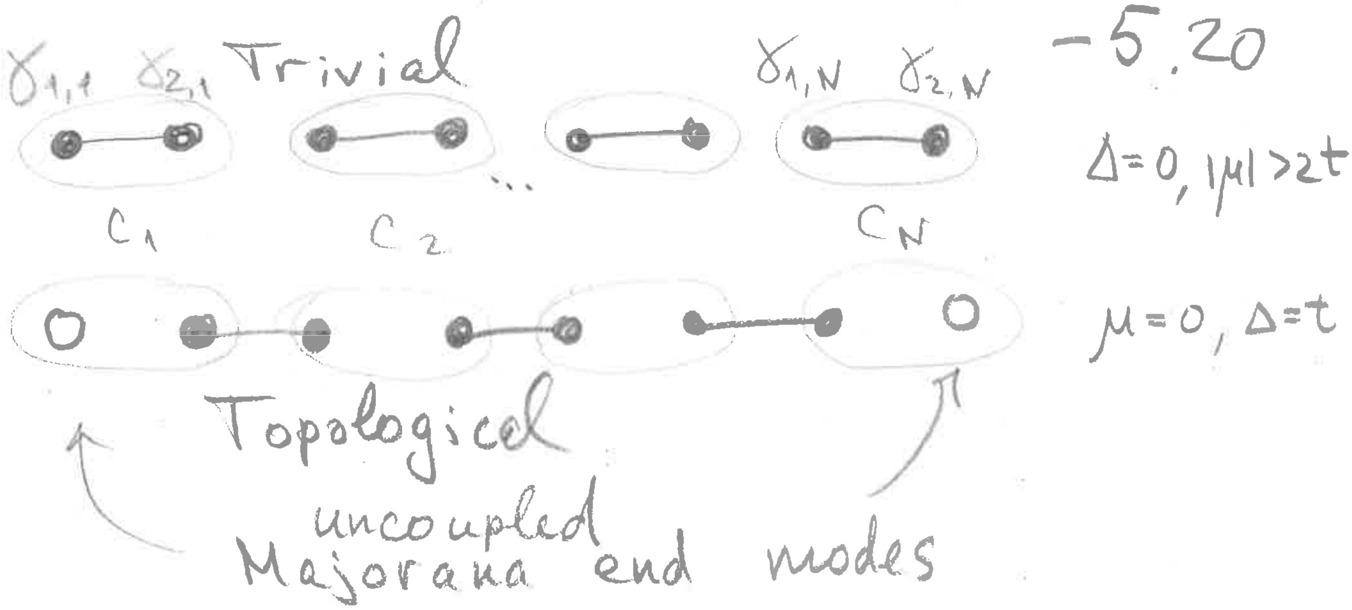



Phase diagram.

Gap closes at transition lines.

I and II : topo with edge modes

III : trivial




Fermionic parity of the ground state.

In addition to $W = 0, \pm 1$, the ground state can be characterized by the fermionic parity (even or odd fermionic number). The number of fermions is not conserved, but the parity is conserved.

Vacuum states of operators $c$:

$$c_j |0\rangle_j = 0$$

Two states with different fermionic parities:

$$|G_e\rangle = \left(\prod_{j=1}^{N-1} d_j d_j^+\right)|0\rangle$$

$$|G_o\rangle = \left(\prod_{j=1}^{N-1} d_j d_j^+\right) c_N^+ |0\rangle$$

$$|0\rangle = |0\rangle_N \ldots |0\rangle_1$$

Any operator $d_j$ annihilates $|G_e\rangle$, $|G_o\rangle$

Topological phase, $\left|\frac{t}{\mu}\right| > \frac{1}{2}$:

robust Majorana edge modes protected by $\mathbb{Z}_2$ fermionic parity.

Define $$\gamma_\Sigma \equiv \sum_{j=1}^{\infty} \left(-\frac{\mu}{2t}\right)^{j-1} \gamma_j$$

$[\gamma_\Sigma, H] = 0 \Rightarrow$ if $|G\rangle$ is a ground state, then $\gamma_\Sigma |G\rangle$ is also a ground state.

These two GSs have opposite parity $P_f = (-1)^{\sum_j n_j} = e^{i\pi \sum_j n_j}$

$\Rightarrow$ linearly independent

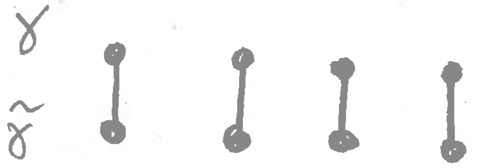
Ground state is twofold degenerate with boundaries; unique for PBC.

For $\left|\frac{t}{\mu}\right| < \frac{1}{2}$, the phase is trivial, has a unique GS independent of boundary conditions.

The two phases cannot be connected without closing the energy gap $\left(\left|\frac{t}{\mu}\right| = \frac{1}{2}\right)$.

Bulk topology cannot be probed by local order parameters but can be identified by nonlocal ones.

III

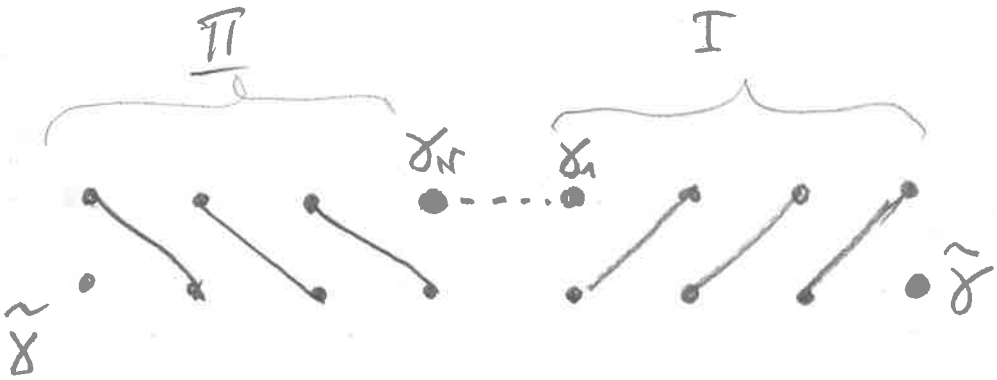



trivial

I


topo

II


coupling $\gamma\gamma$ breaks chirality $(\gamma\tilde\gamma)$
and, equivalently, TRS

TRS broken – $\mathbb{Z}_2$ (trivial, topo)
class D

TRS preserved – $\mathbb{Z}$ $\left(\text{trivial}, \underbrace{\text{topo}_{I}, \text{topo}_{II}}_{\text{distinct}}\right)$

class BDI

# Symmetry fractionalization

Infinite system or PBC - unique GS; symmetry acts trivially in the bulk.

$$H_K = it \sum_j \tilde{\gamma}_j \gamma_{j+1} \qquad c_j^\dagger c_j \; \| $$

Introduce $Q = \prod_j \underbrace{(1-2n_j)}_{\substack{=1 \text{ for empty} \\ =-1 \text{ for occupied}}} = \prod_j (i\tilde{\gamma}_j \gamma_{j+1})$

GS subspace for topo: $\tilde{\gamma}_j \gamma_{j+1} = i$

Trivial phase: $\gamma_j \tilde{\gamma}_j = -i$

$\Rightarrow$ also counts parity

For open boundary conditions, effectively $Q = Q_L Q_R = i \gamma_1 \tilde{\gamma}_N$

$\{Q_L, Q_R\} = 0$ local operators at left and right ends

$\Rightarrow$ protected twofold degeneracy with zero-energy end modes ($L \to \infty$)

Symmetry of the bulk Hamiltonian "fractionalizes" at the edges:

$U = U_L U_R$ with $[U_L, H] = [U_R, H] = 0$.

## Spin example: Cluster model

$$H_C = -\sum_n X_{n-1} Z_n X_{n+1}, \qquad \text{even } N$$

$\sigma_x \quad \sigma_z \quad \sigma_x$

SPT phase protected by $\mathbb{Z}_2 \times \mathbb{Z}_2$:

$$P_1 = Z_1 Z_3 Z_5 \ldots Z_{N-1}$$
$$P_2 = Z_2 Z_4 Z_6 \ldots Z_N$$

$$[H, P_{1,2}] = 0$$

$H_c$: all terms square to 1,
such that eigenvalues are $\pm 1$;
all terms commute
$\Rightarrow$ ground-state subspace:
$X_{n-1} Z_n X_{n+1} = 1$ for all $n$.

This implies
$$X_1 Z_2 Z_4 Z_6 \dots Z_{N-2} X_{N-1} = 1$$
$$\Rightarrow \quad P_2 = X_1 X_{N-1} Z_N$$

Despite $P_2$ is global, in the GS subspace
$$P_2 = \underbrace{P_2^L}_{= X_1} \underbrace{P_2^R}_{= X_{N-1} Z_N}$$

Similarly,
$$P_1 = \underbrace{P_1^L}_{= Z_1 X_2} \underbrace{P_1^R}_{= X_N}$$

$P_1^L$ and $P_2^L$ are anticommuting symmetries
$\Rightarrow$ degenerate edge modes

Adding terms respecting $\mathbb{Z}_2 \times \mathbb{Z}_2$ of $P_1, P_2$ will alter the form of $P_1^L$ and $P_2^L$, but cannot change their anticommutation.

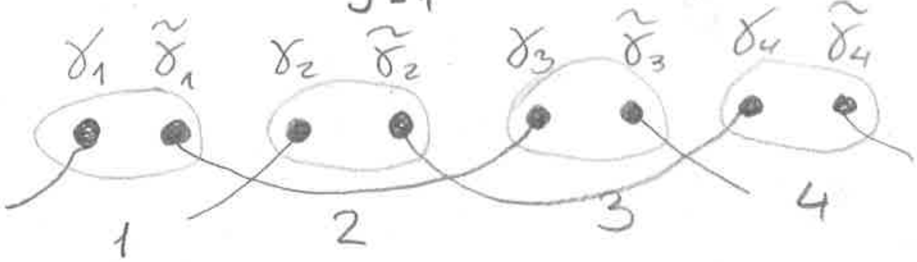
From $P_1 P_2 = P_2 P_1 \Rightarrow P_1^L P_2^L = e^{i\alpha} P_2^L P_1^L$,
$P_1^2 = 1 \Rightarrow e^{i\alpha} = \pm 1$, edges have well-defined degeneracy.

# General Kitaev Chains

Majorana operators $\gamma_j = c_j + c_j^+$

$i\tilde{\gamma}_j = c_j - c_j^+$

$$H_\alpha = i \sum_{j=1}^{N-\alpha} \tilde{\gamma}_j \gamma_{j+\alpha}$$




$\alpha=2$ chain

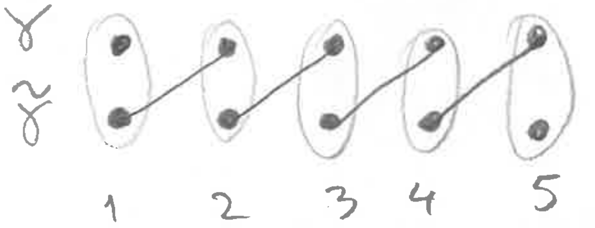
$\alpha=0$ is trivial: spinless band insulator

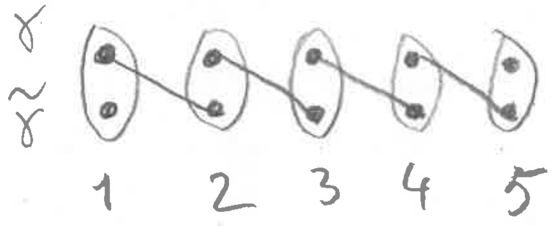
$\alpha=1$ – Kitaev Majorana chain

$\alpha=2$: two Majoranas on each edge (two decoupled Majorana $\alpha=1$ chains)

$\mathbb{Z}$ Parity symmetry: topo vs. trivial, $\mathbb{Z}_2$-classification (class D).
Allowing local terms may break symmetries; chiral symmetry ($\Leftrightarrow$ TRS) is important
TRS $\Rightarrow$ $\mathbb{Z}$ symmetry (BDI class, will be discussed later – classification of topological insulators and superconductors)

$\alpha=1$ Kitaev chain: two topo phases I, II for $\Delta=t$ and $\Delta=-t$.



Use again parity operator

$$Q = \prod_j (i\tilde{\gamma}_j \gamma_j)$$

and rewrite it as follows

$$Q = i^\alpha \left[\prod_{j=1}^{\alpha} \gamma_j\right] \underbrace{\left(\prod_{j=1}^{N-\alpha} i\tilde{\gamma}_j \gamma_{j+\alpha}\right)}_{(-1)^{N-\alpha} \text{ in the GS}} \left[\prod_{j=N-\alpha+1}^{N} \tilde{\gamma}_j\right]$$

Symmetry fractionalization

$$Q = (-1)^{N-\alpha} Q_L Q_R$$

$$Q_L Q_R = (-1)^\alpha Q_R Q_L \rightarrow \text{two distinct possibilities: } \alpha \text{ odd or even}$$

$\alpha = 2$ : $i\gamma_1\gamma_2$ and $i\tilde{\gamma}_{N-1}\tilde{\gamma}_N$ would move edge modes to finite energy but these terms are forbidden by TRS : $T = K$ for spinless fermions

$$Tc_jT = c_j \Rightarrow T\gamma_jT = \gamma_j$$
$$T\tilde{\gamma}_jT = -\tilde{\gamma}_j$$

$$T(i\tilde{\gamma}_{N-1}\tilde{\gamma}_N)T = -i\tilde{\gamma}_{N-1}\tilde{\gamma}_N$$ forbidden

then by parity

$\Rightarrow$ with TRS GS degeneracy is four-fold (one complex fermion at each edge)

With interactions for $\alpha = 8$ degeneracy is lifted $\Rightarrow \mathbb{Z}_8$ classification

Return to SSH chain

$$H_{SSH} = 2(1-\lambda)\sum_j \left(c^\dagger_{A,j} c_{B,j} + H.c\right) + 2\lambda \sum_j \left(c^\dagger_{A,j+1} c_{B,j} + H.c\right)$$

$$2c_{\alpha,j} = \gamma_{\alpha,j} + i\tilde{\gamma}_{\alpha,j}, \qquad \alpha \in \{A, B\}$$

relabel Majoranas:

$$\gamma_{A,j} \equiv \gamma_{2j}, \quad \tilde{\gamma}_{A,j} \equiv \gamma_{2j-1}, \quad \gamma_{B,j} \equiv \tilde{\gamma}_{2j-1}, \quad \tilde{\gamma}_{B,j} \equiv -\tilde{\gamma}_{2j}$$

$$\Rightarrow \quad H_{SSH} = \underbrace{(1-\lambda)\sum_j \left(i\tilde{\gamma}_j \gamma_j\right)}_{0\text{-chain}} + \underbrace{\lambda \sum_j \left(i\tilde{\gamma}_j \gamma_{j+2}\right)}_{2\text{-chain}}$$

$\lambda = 1 \rightarrow$ topological SSH chain with 2 × Majorana = Fermion end states at each end

$\lambda = 0 \rightarrow$ trivial insulator

T and PT symmetries of $\alpha$-chains are equivalent to PH and Chiral (sublattice) symmetries of SSH model

4-chain $\Longleftrightarrow$ AKLT chain (Haldane phase) to be considered below

Symmetry fractionalization: bulk symmetry → projective representation at the edges, discrete phases

# 5.4. Spin Chains, Haldane phase, AKLT

Large spin $S \gg 1$: $H = J \sum_{\langle mn \rangle} \vec{S}_m \vec{S}_n$, $J > 0$

antiferromagnetic chain has long-wavelength excitations (spin waves) with $\epsilon(p) \sim v_s |p|$, as obtained from semiclassical expansion in $1/S \ll 1$. Interactions between spin waves? $S = 1/2, 1, \ldots$?

$S = 1/2$: AF Heisenberg model is equivalent to one-dimensional chiral fermions, $\epsilon(p) = v|p|$ still holds.

However, $S = 1$ chains have gap!

Haldane's conjecture:
spin chains of integer $S$ are gapful (disordered phase with short-range correlations), while chains with half-integer $S$ remain in gapless phase.

Physics depending on parity (2S even or odd)
$\Rightarrow$ hint at topological origin.

Classically: spin ~ sphere of radius S
Space-time (1+1) ~ sphere (after compactification)

$$S^2 \to S^2, \quad \pi_2(S^2) = \mathbb{Z}$$

Single spin (addressed in exercise 2):

$$Z^{(1)} = \int D\vec{n}\, e^{iS\int_0^\beta d\tau\, L_{WZ}}$$

Wess-Zumino Lagrangian

$$L_{WZ} = (1-\cos\theta)\dot{\varphi}, \qquad \theta \text{ and } \varphi \text{ parametrize } \vec{n}$$

(does not admit a globally invariant representation in terms of $\vec{n}$ and $\partial_\tau \vec{n}$; we need an explicit coordinate representation)

Interaction between neighboring spins in the AF Heisenberg chain:

$$J\vec{S}_j \cdot \vec{S}_{j+1} \to JS^2 \vec{n}_j \cdot \vec{n}_{j+1} \to \frac{JS^2}{2}(\vec{n}_j + \vec{n}_{j+1})^2$$

(up to unimportant constants, as $\vec{n}^2 = 1$)

Add WZ terms:

$$S[\vec{n}] = \int d\tau \sum_j \left[\frac{JS^2}{2}(\vec{n}_j + \vec{n}_{j+1})^2 + iS\, L_{WZ}\right]$$

AF spin chain:
Néel Ansatz: $\vec{n}_j = (-1)^j \vec{n}_j'$
Omit prime for brevity

Global rotational invariance

$\vec{n} \to R\vec{n}$, where $R \in O(3)$

$$S_0[\vec{n}] \to \frac{S}{4}\int d^2x\, \partial_{x_\mu}\vec{n}\cdot\partial_{x_\mu}\vec{n}$$

$x, \tau$ rescaled

action of $O(3)$ nonlinear $\sigma$-model

Continuum version of the WZ action

$$S_{top}[\vec{n}] = iS\sum_j (-1)^j \int_0^\beta d\tau\, \Gamma[n_j],$$

where $\Gamma[\vec{n}]$ is the oriented area on the sphere swept by $\vec{n}(\tau)$:


$$\Gamma[-\vec{n}] = 4\pi - \Gamma[\vec{n}]$$
$$= -\Gamma[\vec{n}] \bmod 4\pi$$

For neighboring configurations,

$|\vec{n}_{j+1} - \vec{n}_j| \ll 1$, the area difference

$$\Gamma[\vec{n}_{j+1}] - \Gamma[\vec{n}_j] \simeq \int_0^\beta d\tau\, \vec{n}_j\cdot[(\vec{n}_{j+1}-\vec{n}_j)\times\partial_\tau\vec{n}_j]$$

$$\Rightarrow S_{top}[\vec{n}] \simeq i\frac{S}{2}\int d\tau dx\, \vec{n}\cdot[\partial_x\vec{n}\times\partial_\tau\vec{n}]$$

$\theta$-term with $\theta = \frac{S}{2}$ (cf. Problem set 1)

$$Z = \sum_{Q\in\mathbb{Z}} \int Dn_Q\, e^{2\pi i S Q} e^{-S_0[n_Q]}$$

$Q$ - topo charge, $\Leftarrow$ sum over topo sectors

For integer spin, $e^{2\pi i S Q} = 1$, and the topological term is immaterial.
For half-integer spin, $e^{2\pi i S Q} = (-1)^Q$; different topo-sectors contribute with alternating signs.
Only parity of $2S$ matters.
Difference between $2S$ odd and even is of topological origin.
Fluctuations for integer spin open the gap; for half-integer spin, compensation occurs.
In fact, in the strong-coupling regime
O(3) sigma model $\Rightarrow$ SU(2) WZ theory

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Spin 1: gapful phase
Symmetry $\Longrightarrow$ SPT phase
To understand the "next-level" topology of the Haldane phase, introduce the higher-order terms to the Heisenberg AF model:

$$H = J \sum_j \left[ \vec{S}_j \cdot \vec{S}_{j+1} + \mathcal{D} \left( \vec{S}_j \cdot \vec{S}_{j+1} \right)^2 \right]$$

$\mathcal{D} = \frac{1}{3}$ : Affleck-Kennedy-Lieb-Tasaki model.

AKLT model:

$$H_{AKLT} = J \sum_j \left[ \vec{S}_j \cdot \vec{S}_{j+1} + \frac{1}{3} (\vec{S}_j \cdot \vec{S}_{j+1})^2 \right]$$

Ground state can be found exactly
("valence band state")
Haldane's conjecture tested explicitly

The AKLT Hamiltonian can be rewritten in terms of local projectors:

$$H_{AKLT} = 2J \cdot \sum_j \underbrace{P^{(2)}_{j,j+1}}_{} + \text{const}$$

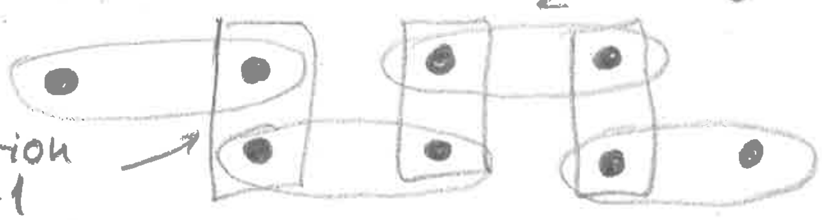
projector onto total spin 2 for neighboring sites,

If each projector $P_i$ individually annihilates the state $|0\rangle$, $P_i|0\rangle = 0$ for all $i$
$E = 0$ (ground state) → "parent Hamiltonian"
Hamiltonians written in terms of local projectors with positive coefficients
≡ "frustration free";

Spin-$\frac{1}{2}$: $H = J \sum_{i=1}^{N} P^{(0)}_{i,i+1} + \text{const.}$ FM.

$P^{(0)}_{i,i+1} = \frac{1}{4} - \vec{S}_i \cdot \vec{S}_{i+1}$ annihilates FM GS.

Represent each spin 1 in the AKLT model by two spins ½:




Similar to Kitaev chain, now with spins. (spin-$\frac{1}{2}$ $\Leftrightarrow$ Majoranas)

Open boundary conditions: 4-fold degeneracy (end spins $\frac{1}{2}$)

PBC: unique GS.

Remark: Matrix-product state

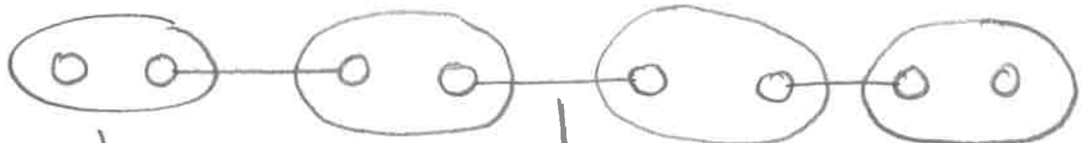
$$|\Psi_{AKLT}\rangle = \sum_{\alpha_i \in \{0,\pm1\}} \left[ \mathrm{Tr}\left(\prod_{i=1}^{L} A_{\alpha_i}\right) |\alpha_1 \ldots \alpha_N\rangle \right]$$

$\alpha_i = 0, \pm 1$ denote z-component of $S_i$

$$A_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_0 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$2\times2$ matrices (bond dimension $\chi = 2$)

any $\alpha_i = \pm 1$ is followed by $\alpha_j = \mp 1$ with $\alpha_{i<k<j} = 0$, successive identical $\alpha = \pm 1$ are forbidden (exercise)



projection to $|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

Since out of four spin-$\frac{1}{2}$ two have total spin 0, the total spin of the four can only be either 0 or 1

$$|\Psi_{AKLT}\rangle = \prod_i P_i^{(1)} \prod_i \frac{1}{\sqrt{2}}\left(|\uparrow_{r,i}\downarrow_{e,i+1}\rangle - |\downarrow_{r,i}\uparrow_{e,i+1}\rangle\right)$$

$$P_i^{(1)} = |{+1}\rangle\langle\uparrow_{e,i}\uparrow_{r,i}| + |{-1}\rangle\langle\downarrow_{e,i}\downarrow_{r,i}| + \frac{1}{\sqrt{2}}|0\rangle\left(\langle\uparrow_{e,i}\downarrow_{r,i}| + \langle\downarrow_{e,i}\uparrow_{r,i}|\right)$$

$|-1\rangle, |0\rangle, |+1\rangle$ – spin-1 representation in $z$-basis

Exponential decay of correlator of local operators $\langle O_i O_j\rangle - \langle O_i\rangle\langle O_j\rangle$

$$\sim e^{-|i-j|/\xi}, \quad \xi = \frac{1}{\ln 3}$$

→ exercise

Closed ring: all spin-$\frac{1}{2}$ are paired
GS is unique
invariant under spin rotation

$$\prod_i e^{i\vec{\Theta}\vec{S}_i} \quad \text{and TRS} \quad \prod_i e^{i\pi S_i^y} K$$

TRS: spin-$\frac{1}{2}$ $\quad T^2 = e^{i\pi\sigma^y} K e^{i\pi\sigma^y} K = -1$

spin-1 $\quad T^2 = e^{i\pi S^y} K e^{i\pi S^y} K = 1$

In $S^z$ basis $+1$ and $-1$ alternate 

String order parameter

$$S_{i,i+r} = S_i^z \left( \prod_{j=i+1}^{i+r-1} e^{i\pi S_j^z} \right) S_{i+r}^z$$

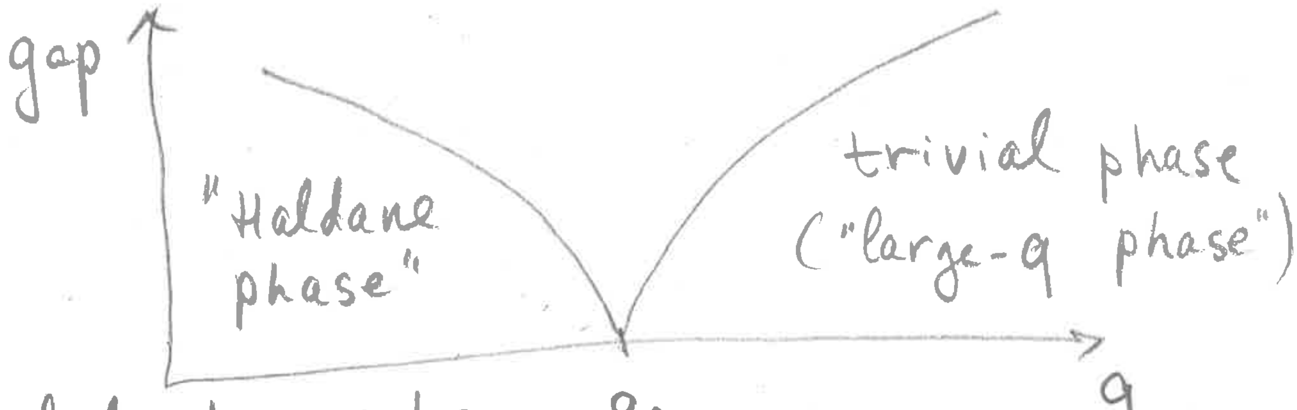
AKLT: $\lim_{r\to\infty} S_{i,i+r} = -\frac{4}{9}$ (constant)

Gap: $\Delta_{AKLT} = 0.35\, J$

All correlations of local operators decay exponentially

"Haldane phase"

$$H = J \sum_j \left( \vec{S}_j \cdot \vec{S}_{j+1} + q \left( S_j^z \right)^2 \right)$$




nonlocal string order parameter

$|\ldots 0000 \ldots \rangle$

AKLT belongs to Haldane phase, cannot be adiabatically transformed to trivial phase without closing the gap