1) a) $n>1$ n und $n+1$ teilfremd $\Rightarrow N_1 = n(n+1)$ hat am mindestens 2 verschiedene Primzahlteiler

$N_2 = N_1(N_1+1) = n(n+1)(n(n+1)+1)$ hat am mindestens 3 verschiedene Primzahlteiler

wir können diesen Schritt unendlichmal machen

b)

n	2^n	n^2	Aussage
1	2	1	✓
2	4	4	✓
3	8	9	X
4	16	16	✓
5	32	25	✓

Induktionsanfang:
$n_0 = 3$, $2^4 \geq 4^2$

Induktionsvoraussetzung:
$2^n \geq n^2$

Induktionsschluss:

$$(n+1)^2 = n^2 + 2n + 1$$

$$n^2 \geq 2n+1 \Rightarrow n^2 - 2n - 1 = (n-1)^2 - 2 \underset{\substack{\uparrow \\ n_0=4 \quad n \geq n_0}}{\geq} 7 > 0$$

$$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot n^2 \geq n^2 + 2n + 1 = (n+1)^2$$

c) (IA): $n_0 = 1$
$8-3=5 \quad 5|5$

(IV): $5|8^n - 3^n$

(IS):

$$8^{n+1} - 3^{n+1} = 8^{n+1} + 8 \cdot 3^n - 3^{n+1} - 8 \cdot 3^n = 8 \cdot \underbrace{(8^n - 3^n)}_{5|8^n-3^n \text{ IV}} + 3^n \underbrace{(8-3)}_{5|5} \Rightarrow 5|8^{n+1} - 3^{n+1}$$

d)
$$(a+b)^s = \frac{a+b}{(a+b)^{1-s}} = \frac{a}{(a+b)^{1-s}} + \frac{b}{(a+b)^{1-s}} \leq \frac{a}{a^{1-s}} + \frac{b}{b^{1-s}} = a^s + b^s$$

2) a) $1+i=\sqrt{2}\exp\left(i\frac{\pi}{4}\right)$

$(1+i)\exp\left(i\frac{\pi}{3}\right)=\sqrt{2}\exp\left(i\left(\frac{\pi}{4}+\frac{\pi}{3}\right)\right)=(1+i)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=$

$=(1+i)\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{1}{2}-\frac{\sqrt{3}}{2}+i\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right)$

$\exp\left(4+i\frac{\pi}{3}\right)\exp\left(-2+i\frac{11\pi}{6}\right)=\exp\left(2+i\frac{13\pi}{6}\right)=\exp\left(2+\frac{\pi}{6}i\right)=e^2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=$

$=e^2\frac{\sqrt{3}}{2}+ie^2\frac{1}{2}$

b) $w=x+ix$

$(1+i)^2=2i$ $\qquad (1+i)^2(3-i)=6i+2$

$x^3(1+i)^3-(3-i)(1+i)^2x^2-i(1+i)x+1+3i=0$ $\qquad (1+i)^3=2i-2$

Re: $1+x-2x^2-2x^3=0$

Im: $3-x-6x^2+2x^3=0$ $\Big\}\Rightarrow 4-8x^2=0\Rightarrow x=\pm\frac{1}{\sqrt{2}}$

$w_1=\frac{1+i}{\sqrt{2}}$

$w_2=-\frac{1+i}{\sqrt{2}}$ $\qquad (w-w_1)(w-w_2)=w^2-i$

$\left(w^3-(3-i)w^2-iw+1+3i\right):\left(w^2-i\right)=w-(3-i)$ ✓

$-(3-i)w^2 \qquad +1+3i$

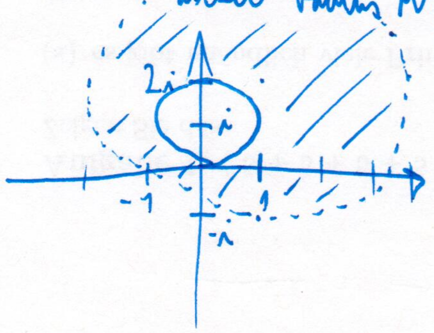
0

$\Downarrow$

$w_3=(3-i)$

c) i) $|w-w_0|=r$

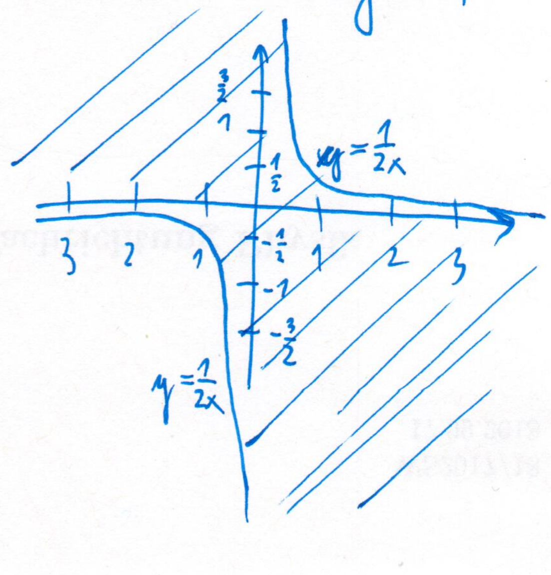
Zirkel Radius r, Mittelpunkt w_0



ii) $w=x+iy$

$w^2=x^2-y^2+2ixy$

$\operatorname{Im} w^2\le 1\Leftrightarrow 2xy\le 1\Rightarrow x\le\frac{1}{2y}$



3) a) $0<a<1$:

$h_n(x)=\sqrt[n]{n^2x}\to 1 \quad n\to\infty$

$h_n\to 1$ punktweise $n\to\infty$

$|h_n(x)-1| = |\sqrt[n]{n^2x}-\sqrt[n]{n^2a}+\sqrt[n]{n^2a}-1| \le |\sqrt[n]{n^2x}-\sqrt[n]{n^2a}| + |\sqrt[n]{n^2a}-1| \overset{a<x\le 1}{=}$

$\le \sqrt[n]{n^2x}-\sqrt[n]{n^2a}+\sqrt[n]{n^2a}-1 \le \sqrt[n]{n^2}-1 =: \alpha_n \to 0 \quad n\to\infty$

h_n gleichmäßig K

$a=0$:

$h_n(0)=0\to 0 \quad n\to\infty$

$h_n(x)\to 1 \quad x\neq 0 \quad n\to\infty$ $\qquad h(x)=\begin{cases}0 & x=0\\ 1 & x\neq 0\end{cases}$

$\Rightarrow$ h nicht stetig $\Rightarrow$ h_n kann nicht gleichmäßig konvergent sein

b) i) $\sum_{n=1}^{\infty} n^2x^{n-1} = s_n \qquad a_n=n^2 \qquad \lim_{n\to\infty}\sqrt[n]{a_n}=1 \Rightarrow$ Konvergenzradius 1

$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x} = h_n \qquad x=\pm 1 \quad \sum_{n=1}^{\infty} n^2(-1)^n \quad a_n\not\to 0 \quad n\to\infty \Rightarrow$

Reihe konvergent für $x\in(-1,1)$

$h_n' = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{1-x}+\frac{x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$(x h_n')' = \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1}{(1-x)^2}+\frac{2x}{(1-x)^3} = \frac{1+x}{(1-x)^3} = s_n$

ii) $\frac{1}{n^2+n-2}\to 0$ + monoton $\to$ Leibnizkriterium

$\frac{1}{n^2+n-2} = \frac{1}{(n+2)(n-1)} = \frac{-\frac{1}{3}}{n+2}+\frac{\frac{1}{3}}{n-1}$

$\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+n-2} = \sum_{n=2}^{\infty}\frac{\frac{1}{3}(-1)^n}{n-1} - \sum_{n=2}^{\infty}\frac{(-1)^n\frac{1}{3}}{n+2} = \frac{1}{3}\left[\sum_{n=2}^{\infty}\frac{(-1)^n}{n-1} - \sum_{n=5}^{\infty}\frac{(-1)^{n+3}}{n-1}\right] =$

$= \frac{1}{3}\left[2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} - \left(1-\frac{1}{2}+\frac{1}{3}\right)\right] = \frac{1}{3}\left[2\ln 2 - \frac{5}{6}\right]$

$\ln(1+x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}x^n$

c) $\sum \frac{1}{n}x^n$ K $[-1,1)$ $\qquad \sum\frac{1}{n}\left(\frac{2}{5}x-\frac{3}{2}\right)^n$ K $[-1,4)$

$\sum\frac{1}{n}\left(\frac{2}{5}x\right)^n$ K $[-\frac{5}{2},\frac{5}{2})$

4) a) $F'(x_0) = \lim\limits_{x\to x_0} \dfrac{F(x)-F(x_0)}{x-x_0}$

i) $\lim\limits_{x\to x_0} \dfrac{\alpha f(x) + \beta g(x) - \alpha f(x_0) - \beta(x_0)}{x-x_0} = \alpha \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0} + \beta \lim\limits_{x\to x_0} \dfrac{g(x)-g(x_0)}{x-x_0}$

$\uparrow$

$\lim a_n + b_n = \lim a_n + \lim b_n$ wenn alle lim existiert

ii) $\lim\limits_{x\to x_0} \dfrac{f(x)\,g(x) - f(x_0)\,g(x_0)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x)\big(g(x)-g(x_0)\big) + g(x_0)\big(f(x)-f(x_0)\big)}{x-x_0}$

$= \lim\limits_{x\to x_0} f(x) \lim\limits_{x\to x_0} \dfrac{g(x)-g(x_0)}{x-x_0} + g(x_0) \lim\limits_{x\to x_0} \dfrac{f(x)-f(x_0)}{x-x_0}$

b) i) $\displaystyle\int_1^\infty \frac{1+\frac{1}{2}(\cos x)^{2018}}{x}\,dx \geq \int_1^\infty \frac{1}{x}\,dx$ "=" $\infty \Rightarrow$ Integral ist divergent

ii) $\displaystyle\int_0^1 \frac{e^x-1}{x^2}\,dx \geq \int_0^\varepsilon \frac{e^x-1}{x^2}\,dx \geq \int_0^\varepsilon \frac{x}{x^2}\,dx = \int_0^\varepsilon \frac{1}{x}\,dx$ "=" $\infty \Rightarrow$ Integral ist divergent

$e^x \geq 1 + x + \dfrac{x^2}{2} \qquad x \geq 0$