1) a) i) $t^{x-1} e^{-\frac{t}{2}} \xrightarrow{t\to 0} 0 \Rightarrow \exists\, t_0 \,\forall t > t_0 : t^{x-1} e^{-\frac{t}{2}} < 1 \Rightarrow t^{x-1} < e^{\frac{t}{2}}$

$$\Gamma(x)=\int_0^\infty t^{x-1} e^{-t}\,dt = \int_0^{t_0} t^{x-1} \underbrace{e^{-t}}_{\le 1} dt + \int_{t_0}^\infty \underbrace{t^{x-1} e^{-t}}_{\le e^{\frac{t}{2}-t} = e^{-\frac{t}{2}}} dt \le \int_0^{t_0} t^{x-1}\,dt + \int_{t_0}^\infty e^{-\frac{t}{2}}\,dt$$

$$\Gamma(x) \le \left[\frac{t^x}{x}\right]_0^{t_0} + \left[-2e^{-\frac{t}{2}}\right]_{t_0}^\infty = \frac{t_0^x}{x} + 2e^{-\frac{t_0}{2}} < \infty \quad \forall x>0$$

ii) INDUKTION:

IA : $n=0 : \Gamma(1) = \int_0^\infty t^0 e^{-t}\,dt = \left[-e^{-t}\right]_0^\infty = 1 = 0!$

IV: $\exists\, n_0 : \Gamma(n_0+1) \stackrel{?}{=} n_0!$

IS: $\Gamma(n_0+1) = n_0! \Rightarrow \Gamma\big((n_0+1)+1\big) = (n_0+1)!$

$$\Gamma(n+2) = \int_0^\infty t^{n+1} e^{-t}\,dt = \begin{bmatrix} f = t^{n+1} \to f' = (n+1)t^n \\ g' = e^{-t} \leftarrow g = -e^{-t} \end{bmatrix} = \left[-t^{n+1}e^{-t}\right]_0^\infty + (n+1)\int_0^\infty t^n e^{-t}\,dt$$

$$= (n+1)\Gamma(n+1) \stackrel{IV}{=} (n+1)\,n! = (n+1)!$$

b) $x \neq 0$: $f'(x) = n x^{n-1} \sin(x^{-m}) + x^n(-m)x^{-m-1}\cos(x^{-m}) = g(x)$

$x=0$ : Existenz $\quad f'(0) \stackrel{?}{=} \lim_{x\to 0} \frac{x^n \sin(x^{-m})}{x} = \lim_{x\to 0} x^{n-1}\sin(x^{-m}) = 0 \quad n-1>0$

Stetigkeit $\quad 0 \stackrel{?}{=} \lim_{x\to 0_-} g(x) = \lim_{x\to 0_-}\Big(\underbrace{n x^{n-1}\sin(x^{-m})}_{\to 0,\ n-1>0} + \underbrace{x^n(-m)x^{-m-1}\cos(x^{-m})}_{\to 0,\ n-m-1>0}\Big)$

$n > m+1$

$0 \stackrel{?}{=} \lim_{x\to 0_+} g(x) = 0$ für $n > m+1$

2) a) i) $\lim_{n\to\infty} a_n = L \iff \forall \varepsilon > 0 \; \exists N \; \forall n > N : |a_n - L| < \varepsilon$

ii) $C$ Häufungspunkt $\iff \exists (k_n)_n, \; k_n \in \mathbb{N}, \; k_n \to \infty : a_{k_n} \xrightarrow[n\to\infty]{} C$

b) i) Folge: $1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, 1, 2, 3, 4, 5, 6, \ldots$

ii) $a_n = n$

c) i) $\sum_{n=1}^{\infty} (-1)^n a_n = \infty$

$$a_n = \begin{cases} 0 & n = 2k-1 \\ \frac{1}{k} & n = 2k \end{cases}$$

ii) die Folge von i) ist nicht monoton

e) $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x, y : |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

$$\left| \frac{1}{1+x^2} - \frac{1}{1+y^2} \right| = \left| \frac{(x+y)(x-y)}{(1+x^2)(1+y^2)} \right| \le |x-y| \left| \frac{x}{(1+x^2)(1+y^2)} + \frac{y}{(1+x^2)(1+y^2)} \right| \le |x-y| \left( \left| \frac{x}{1+x^2} \right| + \left| \frac{y}{1+y^2} \right| \right)$$

$$\underset{\uparrow}{\le} 2|x-y| \Rightarrow \forall \varepsilon > 0 \; \forall x, y : |x-y| < \frac{\varepsilon}{2} \Rightarrow |f(x) - f(y)| < \varepsilon$$

$0 \le \left| \frac{x}{1+x^2} \right| \le 1 \quad \forall x \in \mathbb{R}$ $\qquad x \in (0,1) \quad \frac{x}{1+x^2} \le x < 1$

$\qquad x \in (1, \infty) \quad \frac{x}{1+x^2} \le \frac{x}{x^2} \le \frac{1}{x} \le 1$

$$\left( \frac{1}{1+x^2} \right)' = -\frac{2x}{(1+x^2)^2} \overset{!}{=} 0 \quad x = 0$$

$$\left. \begin{aligned} f'(x) > 0 \quad x < 0 \\ f'(x) < 0 \quad x > 0 \end{aligned} \right\} \Rightarrow f(0) = 1 \text{ Maximum}$$

$\lim_{x\to\infty} f(x) = 0 \qquad \nexists x \in \mathbb{R} : f(x) = 0 \Rightarrow f$ hat kein Minimum

$\lim_{x\to-\infty} f(x) = 0$

d) Potenzreihe $\Rightarrow$ Konvergenzradius $R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \right| = 1$

$s(x)$ K $x \in (-R, R)$

D $x \in \mathbb{R} \setminus (-R, R)$

$s(1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ K Leibnizkriterium $\frac{1}{n} \searrow 0$

$s(-1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ D harmonische Reihe

$s(x)$ K $x \in (-1, 1]$

3) a) i) $z \in A \Leftrightarrow |z+1+i| = |z-3-3i|$ $\quad z = x+iy \quad x,y \in \mathbb{R}$

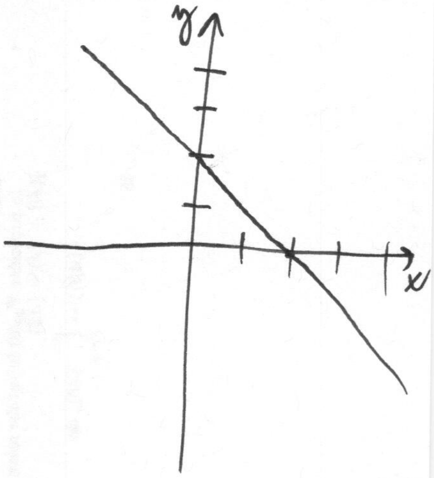
$$|z+1+i|^2 = |z-3-3i|^2$$

$$(x+1)^2 + (y+1)^2 = (x-3)^2 + (y-3)^2$$

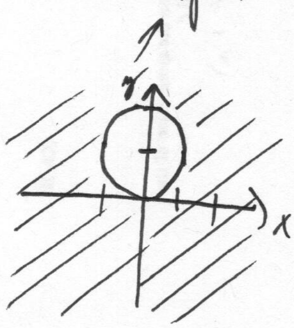
$$x^2+2x+1+y^2+2y+1 = x^2-6x+9+y^2-6y+9$$

$$8x-16 = -8y$$

$$y = -x+2$$



ii) $z \in C \Leftrightarrow x^2 + (y-1)^2 \geq 1 \ \wedge \ (x-1)^2 + (y-2)^2 < 9$ $\quad z = x+iy \quad x,y \in \mathbb{R}$



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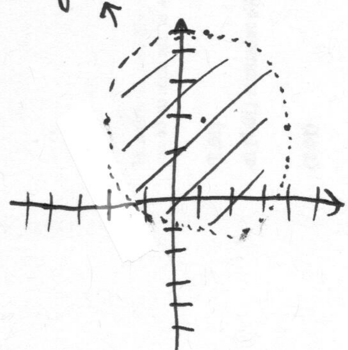
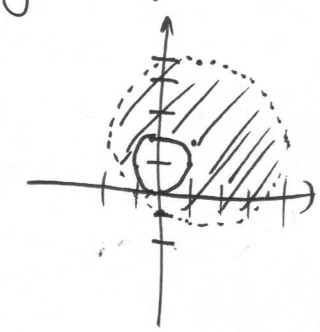


$\Leftrightarrow$



b) Stetige Funktion an beschränkte Intervall nimmt Maximum und Minimum an.

i) $x<3 \quad f(x) = -6x + (3-x+2)^2 \quad f'(x) = -6+2x-10 = -16+2x \quad f'(x) \stackrel{!}{=} 0 \Rightarrow x = 8$

$x>3 \quad f(x) = -6x + (x-3+2)^2 \quad f'(x) = -6+2x-2 = 2x-8 \quad f'(x) \stackrel{!}{=} 0 \Rightarrow x = 4$

| | $x=0$ | $x=3$ | $x=4$ | $x=10$ |
|---|---|---|---|---|
| $f$ | 25 | -14 | -15 | 21 |

Max: 25

Min: -15

ii) $f(x) = \ln(x^2-x+e^{-1}) \quad f'(x) = \dfrac{2x-1}{x^2-x+e^{-1}} \quad f'(x) \stackrel{!}{=} 0 \Rightarrow x = \frac{1}{2}$

| | $x=1$ | $x=3$ |
|---|---|---|
| $f$ | -1 | $\ln(6+e^{-1})$ |

Max: $\ln(6+e^{-1})$

Min: -1

4) a) $\tilde{f}(x) = \ln(1+x)$

$$\tilde{f}(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^5)$$

$$\tilde{g}(x) = \cos x$$

$$\tilde{g}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + o(x^{10}) = 1 + y(x^2) + o(x^{10})$$

$$f(x) = y(x^2) - \frac{(y(x^2))^2}{2} + \frac{(y(x^2))^3}{3} - \frac{(y(x^2))^4}{4} + o(x^{10})$$

$$= \underbrace{\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}\right)}_{y} - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + o(x^{10})$$

$$= x^2\left(-\frac{1}{2!}\right) + x^4\left(\frac{1}{4!} - \frac{1}{2}\left(\left(-\frac{1}{2!}\right)^2\right)\right) + x^6\left(-\frac{1}{6!} - \frac{1}{2}\left(2\cdot\left(-\frac{1}{2!}\right)\frac{1}{4!}\right) + \frac{1}{3}\left(-\frac{1}{2!}\right)^3\right) + x^8\left(\frac{1}{8!} - \frac{1}{2}\left(2\left(-\frac{1}{2!}\right)\left(-\frac{1}{6!}\right) + \left(\frac{1}{4!}\right)^2\right)\right.$$
$$\left. + \frac{1}{3}\left(3\cdot\left(-\frac{1}{2!}\right)^2\frac{1}{4!}\right) - \frac{1}{4}\left(\left(-\frac{1}{2!}\right)^4\right)\right) + o(x^{10})$$

b) i) $\sin(\sqrt{x}) - \sin(\sqrt{x+1}) = \frac{1}{2}\frac{\cos\sqrt{\xi}}{\sqrt{\xi}}(\sqrt{x} - \sqrt{x+1})$ $\qquad f(x) - f(y) = f'(\xi)(x-y) \quad \xi \in (x, y)$

$$= \frac{1}{2}\,\frac{\cos\sqrt{\xi}\;(-1)}{\sqrt{\xi}(\sqrt{x} + \sqrt{x+1})}$$

$$|\sin(\sqrt{x}) - \sin(\sqrt{x+1})| = \left|\frac{1}{2}\,\frac{\cos\sqrt{\xi}\;(-1)}{\sqrt{\xi}(\sqrt{x}+\sqrt{x+1})}\right| \le \frac{1}{2}\,\frac{1}{\sqrt{x}(2\sqrt{x})} \Rightarrow \lim_{x\to\infty}(\sin\sqrt{x} - \sin\sqrt{x+1}) = 0$$

ii) $$\lim_{x\to 0}\frac{\ln(\cos(3x))}{\ln(\cos(2x))} \overset{L'H}{\underset{0/0}{=}} \lim_{x\to 0}\frac{3\frac{\sin 3x}{\cos 3x}}{2\frac{\sin 2x}{\cos 2x}} = \frac{3}{2}\underbrace{\lim_{x\to 0}\frac{\cos 2x}{\cos 3x}}_{1}\lim_{x\to 0}\frac{\sin 3x}{\sin 2x} \overset{L'H}{\underset{0/0}{=}}$$

$$= \frac{3}{2}\lim_{x\to 0}\frac{3\cos 3x}{2\cos 2x} = \frac{9}{4}$$