Auf 1 a) kon. radius $R=\dfrac{1}{\limsup\limits_{n\to\infty}\sqrt[n]{a_n}}$ $\quad s=\sum a_n x^n$

$$R=\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$\lim_{n\to\infty}\left(\frac{n^{-p}}{(n+1)^{-p}}\right)=\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{-p}}=1 \Rightarrow s_p(x)\ D \quad x\notin\langle-1,1\rangle \quad 1p$$

$$s_p(x)\ K \quad x\in(-1,1) \quad 1p$$

$x=-1$: $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^p}$ $\quad$ Leibniz Satz: $\sum a_n(-1)^n \quad a_n\searrow 0 \Rightarrow \sum(-1)^n a_n\ K$

$$\lim_{n\to\infty} n^{-p} = 0 \quad p>0 \qquad n^{-p}\ p>0 \text{ Fallende Folge}$$
$$= 1 \quad p=0$$
$$= \infty \quad p<0 \qquad n^{-p}\ p<0 \text{ Steigende Folge}$$

$\Rightarrow s_p(x)\ K \quad p>0$ Leibniz $\quad 1p$

$s_p(x)\ D \quad p\leq 0$ Nullfolgekriterium $\left(\sum a_n\ K \Rightarrow a_n\to 0\right)$ $1p$

$x=1$: $\sum\limits_{n=1}^{\infty} n^{-p}$

$$\lim_{n\to\infty} n^{-p} = 0 \quad p>0$$
$$\left.\begin{array}{ll} =1 & p=0 \\ =\infty & p<0 \end{array}\right\} \Rightarrow s_p(x)\ D \quad p\leq 0$$

Cauchyverdichtungskriterium: $a_n\searrow 0$: $\quad s=\sum a_n\ K \Leftrightarrow t=\sum\limits_{k=0}^{\infty} a_{2^k}2^k\ K$

$$t=\sum_{k=0}^{\infty}2^k\cdot 2^{-pk}=\sum_{k=0}^{\infty}2^{(1-p)k} \qquad K\Leftrightarrow 2^{1-p}<1 \Leftrightarrow p>1 \quad 1p$$
$$D\Leftrightarrow 2^{1-p}\geq 1 \Leftrightarrow p\leq 1 \quad 1p$$

$$s_p(x)\ K \begin{cases} \langle-1,1\rangle & p>1 \\ \langle-1,1) & p\in(0,1\rangle \\ (-1,1) & p\leq 0 \end{cases}$$

1b) i) $a_n = \sqrt[n]{3^n - \sum_{k=0}^{n} \binom{n}{k} \frac{2^k}{3^{n-k}}}$ $\qquad (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$= \left(3^n - \left(2+\frac{1}{3}\right)^n\right)^{\frac{1}{n}} = \left(3^n - \left(\frac{7}{3}\right)^n\right)^{\frac{1}{n}} =$$

$$= 3\cdot\left(1-\left(\frac{7}{9}\right)^n\right)^{\frac{1}{n}}$$ 1p

$3\sqrt[n]{\frac{1}{2}} \le a_n \le 3 \Rightarrow \lim_{n\to\infty} a_n = 3$ Sandwichsatz 1p

1b) ii) $b_n = \left(1+\frac{1}{n}\right)^{\sqrt{n}} = \left(\left(1+\frac{1}{n}\right)^n\right)^{\frac{1}{\sqrt{n}}}$

$1 \le \left(1+\frac{1}{n}\right)^n \le e$ 1p

$\underset{\substack{\downarrow n\to\infty \\ 1}}{1^{\frac{1}{\sqrt{n}}}} \ge \left(\left(1+\frac{1}{n}\right)^n\right)^{\frac{1}{\sqrt{n}}} \ge \underset{\substack{\downarrow n\to\infty \\ 1}}{e^{\frac{1}{\sqrt{n}}}} \Rightarrow \lim_{n\to\infty} b_n = 1$ Sandwichsatz 1p

2) a) $w^5 = \sqrt{3}i - 1 = 2\left(\frac{\sqrt{3}}{2}i - \frac{1}{2}\right) = 2e^{i\varphi} = 2\exp\left(\left(i\cdot\frac{2\pi}{3} + 2k\pi\right)\right)$ 1p

$\cos\varphi = -\frac{1}{2}$ $\quad \sin\varphi = \frac{\sqrt{3}}{2}$

$\downarrow$

$\varphi = \frac{2}{3}\pi, \varphi = \frac{4\pi}{3}$ $\quad \varphi = \frac{\pi}{3}, \varphi = \frac{2\pi}{3}$

$w_1 = \sqrt[5]{2}\exp\left(i\frac{2\pi}{3\cdot 5}\right)$

$w_2 = \sqrt[5]{2}\exp\left(i\frac{1}{5}\left(\frac{2\pi}{3} + 2\pi\right)\right)$

$w_3 = \sqrt[5]{2}\exp\left(i\frac{1}{5}\left(\frac{2\pi}{3} + 4\pi\right)\right)$

$w_4 = \sqrt[5]{2}\exp\left(i\frac{1}{5}\left(\frac{2\pi}{3} + 6\pi\right)\right)$

$w_5 = \sqrt[5]{2}\exp\left(i\frac{1}{5}\left(\frac{2\pi}{3} + 8\pi\right)\right)$

1p

2b) $\frac{(1+4i)^3}{1-i} = \frac{(1+4i)^3}{(1-i)}\frac{(1+i)}{(1+i)} = \frac{(-15+8i)(-3+5i)}{2} = \frac{5}{2} - \frac{99}{2}i$

$\exp(-1+i) = \frac{\exp i\pi}{e} = -\frac{1}{e}$ 1p

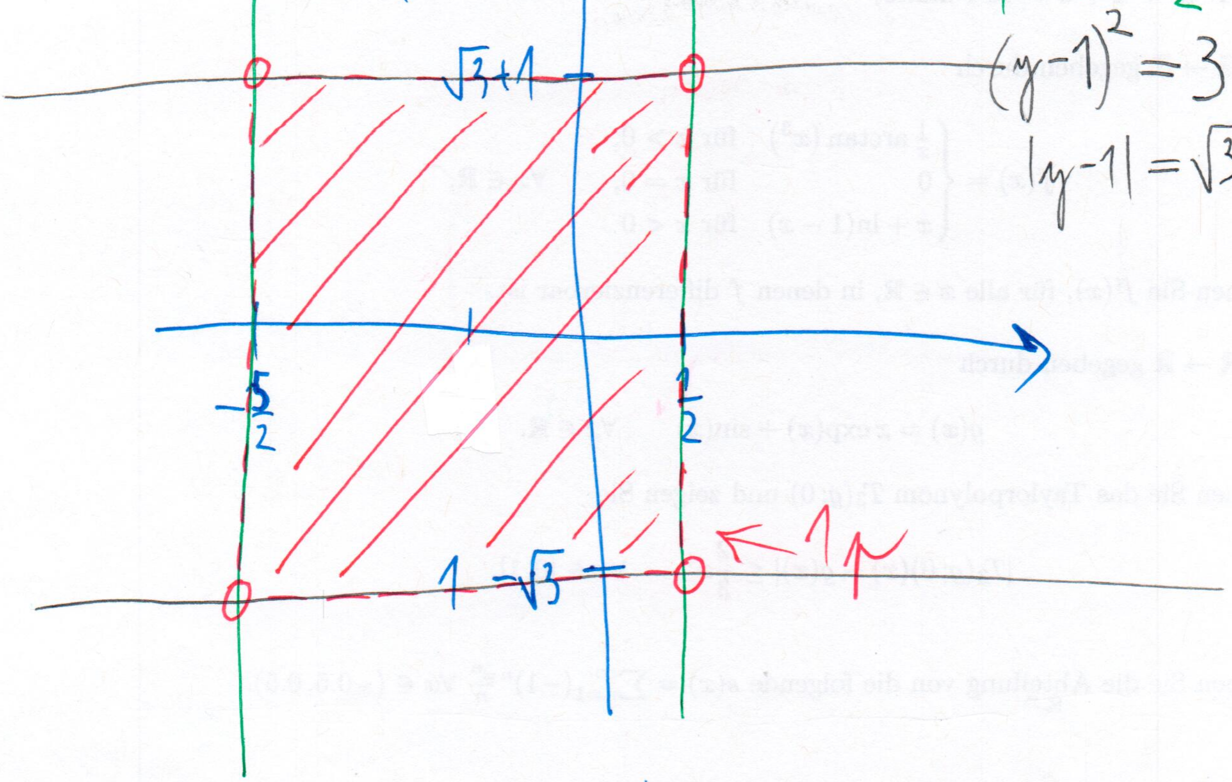
$\frac{(1+4i)^3}{1-i} - \exp(-1+i)(4i-2) = \frac{5}{2} - \frac{99}{2}i - \frac{2}{e} + \frac{4}{e}i = \frac{5}{2} - \frac{e}{2} + i\left(\frac{4}{e} - \frac{99}{2}\right)$

1p

2c) i) $z = x + iy$

$(y-1)^2 \leq 2|x+1| < 3$

$2|x+1| \leq (y-1)^2 < 3$



$|x+1| \leq \frac{3}{2}$ 0,5 P

$(y-1)^2 = 3$

$|y-1| = \sqrt{3}$ 0,5 P

1 P

2c) ii)	$y^2 \leq 2\|x-1\|$	$2\|x-1\| < y^2$
$2\|x\| \leq (y-3)^2$	$2\|x-1\| = (y-3)^2$ (a)	$y^2 = (y-3)^2$ (b)
$(y-3)^2 < 2\|x\|$	$2\|x-1\| = 2\|x\|$ (c)	$2\|x\| = y^2$ (d)

(a) $x > 1$ $\quad 2(x-1) = (y-3)^2 \rightarrow x = \frac{(y-3)^2+2}{2}$ $\quad$ X weil $2|x| \leq (y-3)^2$

$x < 1$ $\quad 1 - x = \frac{(y-3)^2}{2} \rightarrow x = 1 - \frac{(y-3)^2}{2} \rightarrow y^2 \leq (y-3)^2 \rightarrow 6y \leq 9$

$y \leq \frac{3}{2}$

$2|x| \leq (y-3)^2 \quad 0 < x < 1 \rightarrow 2x \leq (y-3)^2$

$2 - (y-3)^2 \leq (y-3)^2 \rightarrow$

$1 \leq (y-3)^2 \rightarrow y \notin (2,4)$

$x < 0 \quad -2x \leq (y-3)^2$

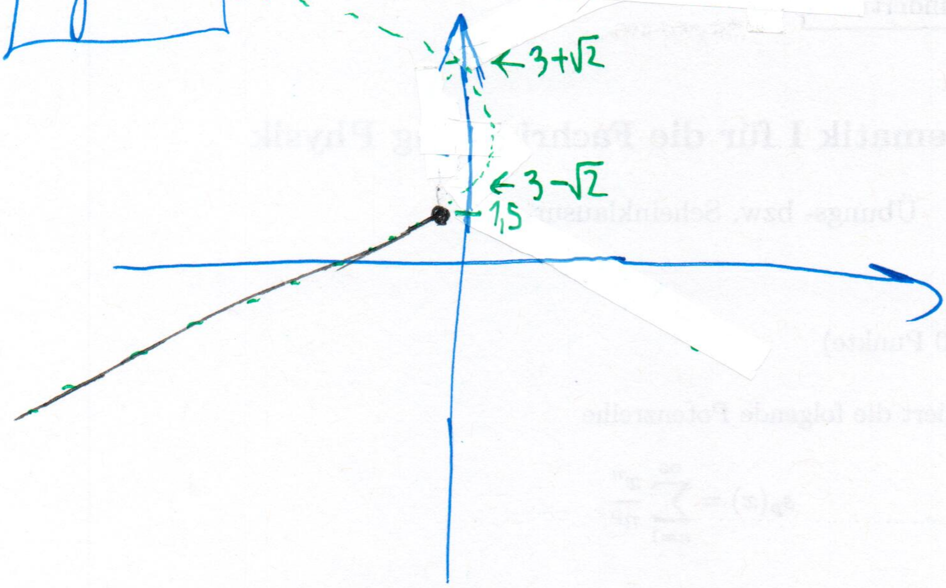
$-2 \leq 0$ ✓

4c) a) $y \leq 2 \leftarrow x \in (0,1)$ $x = 1 - \frac{(y-3)^2}{2}$

$\boxed{y \leq \frac{3}{2}} \quad x < 0$

$x \quad y^2 - 6y + 7 = 0$

$y_1, y_2 = \frac{6 \pm \sqrt{36-28}}{2} = 3 \pm \sqrt{2}$



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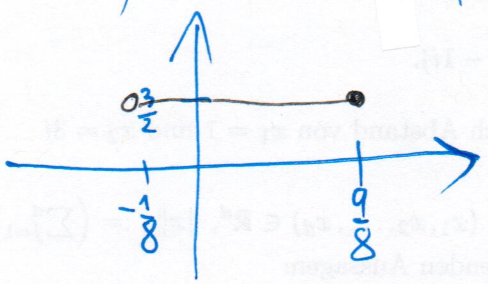
b) $y^2 = (y-3)^2 \Leftrightarrow 6y = 9 \to y = \frac{3}{2}$

$2|x-1| < \frac{9}{4} \quad \wedge \quad 2|x| \leq \frac{9}{4}$

$|x-1| < \frac{9}{8} \quad \wedge \quad |x| \leq \frac{9}{8}$

$x \in \left(-\frac{1}{8}, \frac{17}{8}\right) \quad x \in \left[-\frac{9}{8}, \frac{9}{8}\right] \to y = \frac{3}{2} \quad x \in \left(-\frac{1}{8}, \frac{9}{8}\right]$



1

c) $2|x-1| = 2|x| \to |x-1| = |x| \quad x = \frac{1}{2}$

$(y-3)^2 < 1 \wedge y^2 \leq 1$

$y \in (2,4) \wedge y \in [-1,1]$

kein Punkte

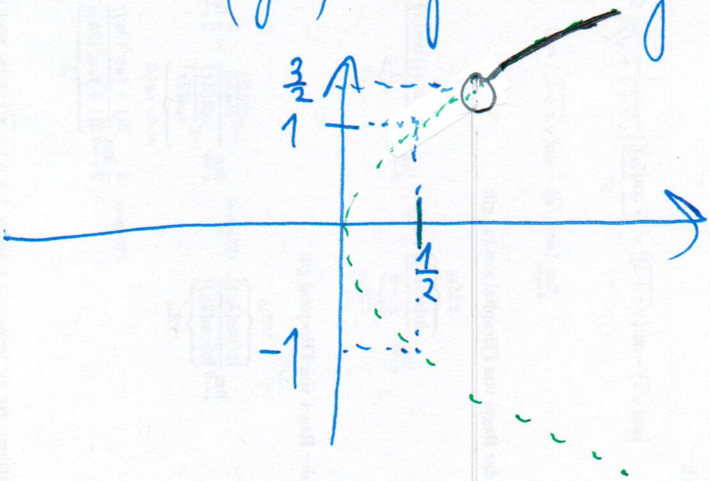
1

d) $2|x| = y^2$ $\quad x \geq 0 \quad 2x = y^2 \quad +$

$\quad x < 0 \quad -2x = y^2 \quad -$

$+: \quad 2|x-1| < y^2 \rightarrow 2|x-1| < 2x \quad |x-1| < x \rightarrow x > \frac{1}{2}$

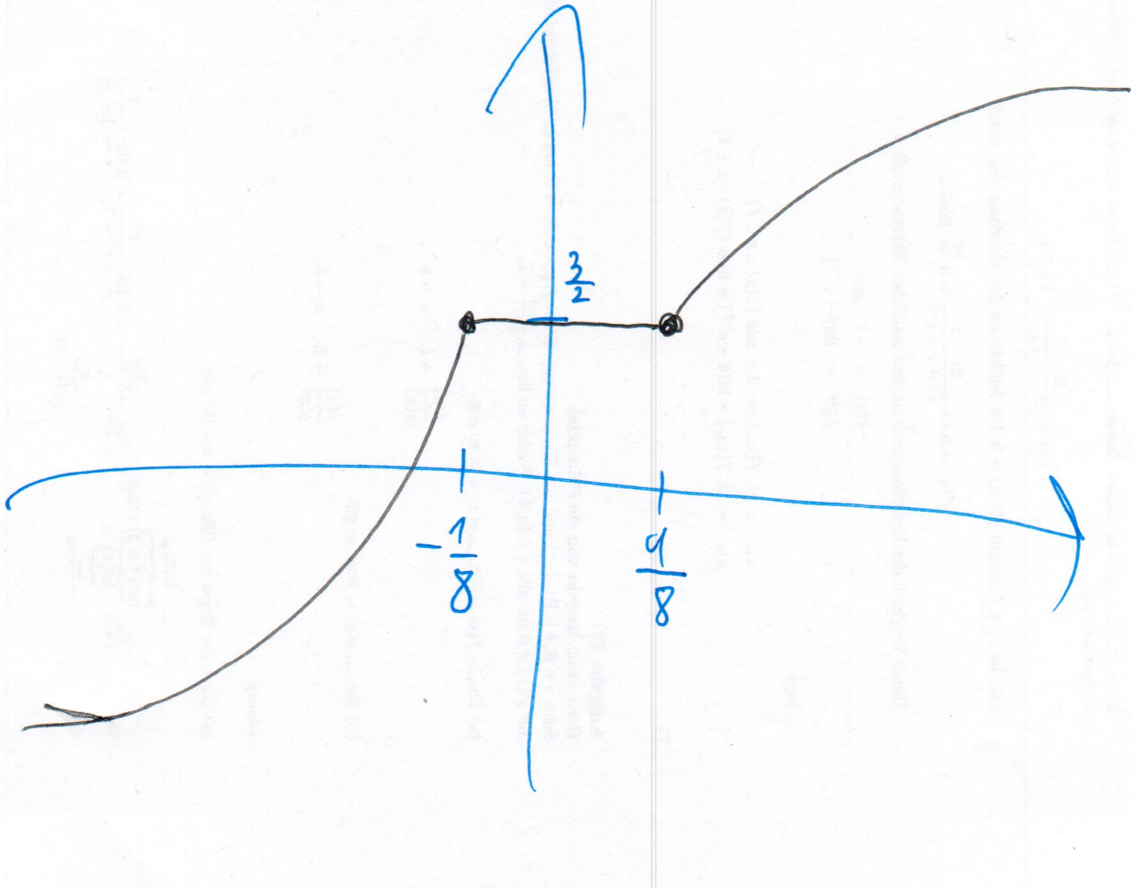
$(y-3)^2 < 2|x| \rightarrow (y-3)^2 < y^2 \rightarrow 9 < 6y \rightarrow \frac{3}{2} < y$



$-: \quad 2|x-1| < -2x \rightarrow 2(1-x) < -2x \rightarrow 2 < 0 \quad \times$

$(y-3)^2 < y^2 \rightarrow \frac{3}{2} < y$ $\quad$ kein Punkt

1



3) Axiom: 1) $\|\alpha x\| = \alpha \|x\|$

2) $\|x\| \geq 0$, $\|x\| = 0 \Rightarrow \vec{x} = \vec{0}$

3) $\|x+y\| \leq \|x\| + \|y\|$

1) $\|\alpha x\|_\infty = \max\limits_{i \in j} \{|\alpha x_i|\} = \max\limits_i \{|\alpha||x_i|\} = |\alpha| \max \{|x|_i\}$

$\|\alpha x\|_p = \left(\sum\limits_i |\alpha x_i|^p\right)^{\frac{1}{p}} = \left(\sum |\alpha|^p |x_i|^p\right)^{\frac{1}{p}} = \left(|\alpha|^p\right)^{\frac{1}{p}} \left(\sum |x_i|^p\right)^{\frac{1}{p}}$ 1p

2) $\|x\| \geq 0$ trivial $|x_i| \geq 0$

$\|x\|_\infty = 0 \Leftrightarrow \max |x_i| = 0 \Rightarrow x_i = 0$

$\|x\|_p = 0 \Leftrightarrow \sum |x_i|^p = 0 \Rightarrow x_i = 0$ 1p

3) $\|x+y\|_\infty = \max\limits_i \{|x_i + y_i|\} \leq \max\limits_i \{|x|_i + |y_i|\} \leq \max\limits_i \{|x_i|\} + \max\limits_i \{|y_i|\}$

1p

$\|x+y\|_p$ $F(x) = x^p$ konvex

$\vec{x} = \vec{0}$ oder $\vec{y} = \vec{0}$ trivial

Konvexität

$|x_j + y_j|^p = (\|\vec{x}\|_p + \|y\|_p)^p \left[(1-\lambda) \frac{x_j}{\|x\|_p + \|y\|_p} + \lambda \frac{y_j}{\|x\|_p + \|y\|_p}\right]^p \leq (\|x\|_p + \|y\|_p)^p \left[(1-\lambda)|X_j|^p + \lambda |Y_j|^p\right]$

$X_j = \frac{x_j}{\|x\|_p}$ $\quad Y_j = \frac{y_j}{\|y\|_p}$ $\quad \frac{x_j}{\|x\|_p + \|y\|_p} \leq \frac{x_j}{\|x\|_p} = X_j$; $\frac{y_j}{\|x\|_p + \|y\|_p} \leq \frac{y_j}{\|y\|_p} = Y_j$

$\lambda = \frac{\|\vec{y}\|_p}{\|\vec{x}\|_p + \|\vec{y}\|_p} \rightarrow \sum\limits_{j=1}^{d} |x_j + y_j|^p = \|\vec{x} + \vec{y}\|_p^p \leq (\|x\|_p + \|y\|_p)^p \sum\limits_{j=1}^{d} \left[(1-\lambda)|X_j|^p + \lambda |Y_j|^p\right]$

$\|\vec{x} + \vec{y}\|_p^p \leq (\|\vec{x}\|_p + \|\vec{y}\|_p)^p \left[(1-\lambda)\|\vec{X}\|^p + \lambda \|\vec{Y}\|^p\right] = (\|\vec{x}\|_p + \|\vec{y}\|_p)^p$

3) $\|\vec{x}+\vec{y}\|_p \leq \|\vec{x}\|_p + \|\vec{y}\|_p$ 3P

b) $$\|x\|_\infty = \max_j |x_j| \underset{\uparrow}{=} \left(\max_j \left(|x_j|^p\right)\right)^{\frac{1}{p}} \leq \left(\sum |x_j|^p\right)^{\frac{1}{p}} = \|x\|_p$$ ½P

$f(x)=x^p$ monoton steigende

$$\|x\|_p = \left(\sum_j |x_j|^p\right)^{\frac{1}{p}} \leq \left(d \max_j\{|x_j|^p\}\right)^{\frac{1}{p}} = d^{\frac{1}{p}} \max_j \{|x_j|\}$$ 1P

c) $$\|x\|_\infty \overset{(2)}{\leq} \|x\|_p \overset{(1)}{\leq} d^{\frac{1}{p}} \|x\|_\infty \quad \Rightarrow$$

$$\|x\|_\infty \overset{(1)}{\leq} \|x\|_q \overset{(2)}{\leq} d^{\frac{1}{q}} \|x\|_\infty$$

½P 1P

$$\Rightarrow \quad \|x\|_p \overset{(1)}{\leq} d^{\frac{1}{p}} \|x\|_q \Leftrightarrow \underset{C_1}{d^{-\frac{1}{p}}} \|x\|_p \leq \|x\|_q \overset{(2)}{\leq} \underset{C_2}{d^{\frac{1}{q}}} \|x\|_p$$

4 a) $x>0$: $f'(x) = -\frac{1}{x^2}\arctan(x^3) + \frac{1}{x}\frac{3x^2}{1+x^6}$ 1

$x<0$: $f'(x) = 1 - \frac{1}{1-x}$ 1

$$f_+'(0) = \lim_{x\to 0_+} \frac{\frac{1}{x}\arctan x^3 - 0}{x} \underset{\text{L'HOPITAL}}{=} \lim_{x\to 0_+}\left(-\frac{1}{x^2}\arctan x^3 + \frac{3x}{1+x^6}\right) \underset{\text{L'HOPITAL}}{=} 0$$ 1

$$f_-'(0) = \lim_{x\to 0_-} \frac{x+\ln(1-x)}{x} = \lim\left(1+\frac{\ln(1-x)}{x}\right) \underset{\text{L'HOPITAL}}{=} 1+\lim\left(\frac{-1}{1-x}\right) = 0$$ 1

b) f stetig $\Rightarrow$ (an $[a,b]$) $\exists$ Maximum + Minimum 1

$$f(x) = \sqrt{3-x+\frac{2}{\pi}\sin(\pi x)}$$

$$f'(x) = \frac{1\,(-1+2\cos(\pi x))}{2\sqrt{3-x+\sin \pi x}}$$ 1

$f'(x) \overset{!}{=} 0 \iff \frac{1}{2} = \cos(\pi x)$

$x = \frac{1}{3}, \frac{5}{3}, \frac{7}{3}$

$f(0) = \sqrt{3}$

$f(3) = 0 \Leftarrow \underline{\min}$

$f\left(\frac{5}{3}\right) = \sqrt{\frac{4}{3} - \frac{2}{\pi}\frac{\sqrt{3}}{2}} = \sqrt{\frac{4}{3} - \frac{\sqrt{3}}{\pi}}$

$f\left(\frac{7}{3}\right) = \sqrt{\frac{2}{3} + \frac{2}{\pi}\frac{\sqrt{3}}{2}} = \sqrt{\frac{2}{3} + \frac{\sqrt{3}}{\pi}}$

$f\left(\frac{1}{3}\right) = \sqrt{\frac{8}{3} + \frac{2}{\pi}\frac{\sqrt{3}}{2}} = \sqrt{\frac{8}{3} + \frac{\sqrt{3}}{\pi}} \leftarrow \max$ weil $\frac{\sqrt{3}}{\pi} \geq \frac{1{,}5}{3{,}2} > \frac{1}{3}$ 1

c) $s(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n} = -\ln(1+x)$ $R = (-1,1)$ 3

$$\boxed{s'(x) = -\frac{1}{1+x}}$$