1) a) $\det(A-\lambda) = \begin{vmatrix} 1-\lambda & -1 \\ -1 & -1-\lambda \end{vmatrix} \cdot \begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} \overset{!}{=} 0 \qquad \det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A \det C$

$0 = \left((1-\lambda)(-1-\lambda)-1\right)(2-\lambda)(1-\lambda) = \left(\lambda^2-1-1\right)(2-\lambda)(1-\lambda) = (2-\lambda)(1-\lambda)(\lambda-\sqrt{2})(\lambda+\sqrt{2})$

$\lambda = -\sqrt{2},\ 1,\ \sqrt{2},\ 2$

m_a: 1 1 1 1

b) $(A+\sqrt{2}\,|\,0) \sim \begin{pmatrix} 1+\sqrt{2} & -1 & 0 & 1 \\ -1 & -1+\sqrt{2} & -1 & 0 \\ 0 & 0 & 2+\sqrt{2} & 3 \\ 0 & 0 & 0 & 1+\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1+\sqrt{2} & -1 & 0 & 0 \\ 0 & (1+\sqrt{2})(-1+\sqrt{2})-1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim$

$\sim \begin{pmatrix} 1+\sqrt{2} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \overline{v_1} = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ 1+\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$m_g(-\sqrt{2}) = \mathrm{Lin}(\overline{v_1})$

$(A-1\,|\,0) \sim \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \quad \overline{v_2} = \frac{1}{\sqrt{12}} \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}$

$m_g(1) = \mathrm{Lin}(\overline{v_2})$

$(A-\sqrt{2}\,|\,0) \sim \begin{pmatrix} 1-\sqrt{2} & -1 & 0 & 1 \\ -1 & -1-\sqrt{2} & -1 & 0 \\ 0 & 0 & 2-\sqrt{2} & 3 \\ 0 & 0 & 0 & 1-\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1-\sqrt{2} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \quad \overline{v_3} = \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ 1-\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$

$m_g(\sqrt{2}) = \mathrm{Lin}(\overline{v_3})$

$(A-2\,|\,0) \sim \begin{pmatrix} -1 & -1 & 0 & 1 \\ -1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow v_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \quad \overline{v_4} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$

$m_g(2) = \mathrm{Lin}(\overline{v_4})$

c) ja, $A = S \begin{pmatrix} -\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix} S^{-1} \qquad S = \left(\overline{v_1}\ \overline{v_2}\ \overline{v_3}\ \overline{v_4}\right)$

2) a) $\|p_1\|^2 = \int_{-\pi}^{\pi} \sin^2 y \, dy = \int_{-\pi}^{\pi} \cos^2 y \, dy \Rightarrow \int_{-\pi}^{\pi} \sin^2 y \, dy = \pi$

$\overline{p}_1 = \frac{1}{\sqrt{\pi}} p_1$

$(\overline{p}_1, p_2) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin^2 y \, dy = \frac{\pi}{\sqrt{\pi}}$

$\tilde{p}_2 = p_2 - (\overline{p}_1, p_2)\overline{p}_1 = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} - \begin{pmatrix} \sin x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \cos x \end{pmatrix}$

$\|\tilde{p}_2\|^2 = \int_{-\pi}^{\pi} \cos^2 y \, dy = \pi$

$\overline{p}_2 = \frac{1}{\sqrt{\pi}} \begin{pmatrix} 0 \\ \cos x \end{pmatrix}$

$(\overline{p}_1, p_3) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \underbrace{-\sin x \cos x}_{\text{ungerade Funk.}} \, dx = 0$

$(\overline{p}_2, p_3) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos x \, dx = 0$

$\|p_3\|^2 = \int_{-\pi}^{\pi} (\cos^2 x + 1) \, dx = 3\pi$

$\overline{p}_3 = \frac{1}{\sqrt{3\pi}} p_3$

$(\overline{p}_1, p_4) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin x \, dx = 0$

$(\overline{p}_2, p_4) = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos x \, dx = 0$

$(\overline{p}_3, p_4) = \frac{1}{\sqrt{3\pi}} \int_{-\pi}^{\pi} -\cos x + 1 \, dx = \frac{2\pi}{\sqrt{3\pi}}$

$\tilde{p}_4 = p_4 - (\overline{p}_3, p_4)\overline{p}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} -\cos x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{2}{3}\cos x \\ \frac{1}{3} \end{pmatrix}$

$\|\tilde{p}_4\| = \int_{-\pi}^{\pi} 1 + \frac{4}{3}\cos x + \frac{4}{9}\cos^2 x + \frac{1}{9} \, dx = \frac{4}{9}\pi + 2\pi + \frac{2\pi}{9} = \frac{24\pi}{9}$

$\overline{p}_4 = \frac{3}{2\sqrt{6\pi}} \tilde{p}_4$

2) b) ${}^{X}\jmath^{Y} = {}^{X}\jmath^{\mathcal{E}_3}\, {}^{\mathcal{E}_3}\jmath^{Y}$

$${}^{\mathcal{E}_3}\jmath^{Y} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\left({}^{X}\jmath^{\mathcal{E}_3}\right)^{-1} = {}^{\mathcal{E}_3}\jmath^{X} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

$${}^{X}\jmath^{\mathcal{E}_3}: \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$${}^{X}\jmath^{Y} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{pmatrix}$$

3) a) $\nabla f = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$

max für $x\uparrow, y\uparrow, z\downarrow$ $\max_W f = 3$

min für $x\downarrow, y\downarrow, z\uparrow$ $\min_W f = -9$

b) $L = 5x + y - 3z - \lambda_1(x+y+z) - \lambda_2(x^2+y^2+z^2-1)$

$\partial_x L = 5 - \lambda_1 - 2\lambda_2 x \overset{!}{=} 0$ (1) $\quad x+y+z=0$ (4)

$\partial_y L = 1 - \lambda_1 - 2\lambda_2 y \overset{!}{=} 0$ (2) $\quad x^2+y^2+z^2=1$ (5)

$\partial_z L = -3 - \lambda_1 - 2\lambda_2 z \overset{!}{=} 0$ (3)

(1+2+3): $3 - 3\lambda_1 - 2\lambda_2(x+y+z) = 0 \Rightarrow \lambda_1 = 1$ (6)

(1+6) $2x\lambda_2 = 4$

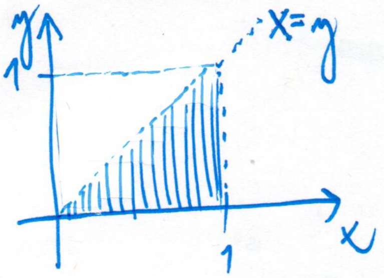
(2+6) $2y\lambda_2 = 0 \Rightarrow \lambda_2 = 0 \vee y = 0$ $\qquad$ (1+6) $\Rightarrow \lambda_2 \neq 0$

(3+6) $2z\lambda_2 = -4$

$x = \frac{2}{\lambda_2}$ $\quad z = -\frac{2}{\lambda_2}$ $\quad$ + (5) $\Rightarrow \frac{4}{\lambda_2^2} + \frac{4}{\lambda_2^2} = 1 \Rightarrow \lambda_2 = \pm\sqrt{8}$

	x	y	z	$f(x,y,z)$	
1 :	$\frac{2}{\sqrt8}$	0	$-\frac{2}{\sqrt8}$	$\frac{16}{\sqrt8} = 2\sqrt8$	$\leftarrow \max_S f$
2 :	$-\frac{2}{\sqrt8}$	0	$\frac{2}{\sqrt8}$	$-\frac{16}{\sqrt8} = -2\sqrt8$	$\leftarrow \min_S f$

4) a)



$$\int_0^1\int_y^1 x^2 e^{x^2}\,dx\,dy = \int_0^1\int_0^x x^2 e^{x^2}\,dy\,dx = \int_0^1 x^3 e^{x^2}\,dx = \begin{bmatrix} x^2 = y \\ x\,dx = \frac{1}{2}dy \end{bmatrix} = \frac{1}{2}\int_0^1 y\,e^y\,dy =$$

$$= \frac{1}{2}\left(\left[y\,e^y\right]_0^1 - \int_0^1 e^y\,dy\right) = \frac{1}{2}$$

b) $$I = \int_B xy + yz + xz\,dx\,dy\,dz = 3\int_B xy\,dx\,dy\,dz = 3\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 r^4 \sin\varphi\cos\varphi\cos^3\theta\,dr\,d\theta\,d\varphi$$

$x = r\cos\varphi\cos\theta$

$y = r\sin\varphi\cos\theta$

$z = r\sin\theta$

$$I = 3\,\frac{1}{5}\,\frac{1}{2}\,\frac{2}{3} = \frac{1}{5}$$

$$\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\,d\varphi = \frac{1}{4}\left[\cos 2\varphi\right]_0^{\frac{\pi}{2}} = \frac{1}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^3\theta\,d\theta = \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\cos\theta\,d\theta = \begin{bmatrix} \sin\theta = y \\ \cos\theta\,d\theta = dy \end{bmatrix} = \int_0^1 (1-y^2)\,dy = \left[1 - \frac{y^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\int_0^1 r^4\,dr = \left[\frac{r^5}{5}\right]_0^1 = \frac{1}{5}$$