Auf 1)

a)

$$\det(A-\lambda) \overset{(1)}{=} \det\begin{pmatrix}1-\lambda & 0\\ -\frac{1}{2} & 2-\lambda\end{pmatrix} \det\begin{pmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{pmatrix} = (1-\lambda)(2-\lambda)(4-5\lambda+\lambda^2+2) =$$

$$= (\lambda-2)(\lambda-1)(\lambda-2)(\lambda-3) \qquad \lambda_A = 1,2,3$$

$m_a(1) = 1$

$m_a(2) = 2$

$m_a(3) = 1$

$$\det\begin{pmatrix}M_1 & M_2\\ 0 & M_3\end{pmatrix} = \det M_1 \det M_3 \quad (1)$$

b)

$$(A-1) = \begin{pmatrix}0 & 0 & -1 & 2\\ -\frac{1}{2} & 1 & 0 & \frac{1}{2}\\ 0 & 0 & 3 & -2\\ 0 & 0 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & -2 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix} \Rightarrow E_A(1) = \mathrm{Lin}\left(\begin{pmatrix}2\\1\\0\\0\end{pmatrix}\right)$$

$m_g(1) = 1$

$$(A-2) = \begin{pmatrix}-1 & 0 & -1 & 2\\ -\frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & 2 & -2\\ 0 & 0 & 1 & -1\end{pmatrix} \sim \begin{pmatrix}-1 & 0 & 1 & 2\\ 0 & 0 & -1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \Rightarrow E_A(2) = \mathrm{Lin}\left(\begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right)$$

$m_g(2) = 2$

$$(A-3) = \begin{pmatrix}-2 & 0 & -1 & 2\\ -\frac{1}{2} & -1 & 0 & \frac{1}{2}\\ 0 & 0 & 1 & -2\\ 0 & 0 & 1 & -2\end{pmatrix} \sim \begin{pmatrix}1 & 2 & 0 & -1\\ 0 & 4 & -1 & 0\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0\end{pmatrix} \Rightarrow E_A(3) = \mathrm{Lin}\left(\begin{pmatrix}0\\ \frac{1}{2}\\ 2\\ 1\end{pmatrix}\right)$$

$m_g(1) = 1$

c) ja $m_a(\lambda_i) = m_g(\lambda_i)$

$A = SDS^*$

$$v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\\0\end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\\1\\1\end{pmatrix} \qquad v_3 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad v_4 = \frac{2}{\sqrt{21}}\begin{pmatrix}0\\ \frac{1}{2}\\ 2\\ 1\end{pmatrix}$$

$$S = \begin{pmatrix}v_1 & v_2 & v_3 & v_4\end{pmatrix}$$

$$D = \mathrm{diag}(1,2,2,3)$$

Auf 2)

1) $$\langle p_0, p_0\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-y^2}}\,dy = \begin{bmatrix} y = \cos x \\ dy = -\sin dx \end{bmatrix} = \int_0^{\pi} \frac{\sin x}{\sin x}\,dx = \pi$$

$$\underline{\bar{p}_0 = \frac{1}{\sqrt{\pi}}}$$

$$\langle \bar{p}_0, p_1\rangle = \int_{-1}^{1} \underbrace{\frac{y\frac{1}{\sqrt{\pi}}}{\sqrt{1-y^2}}}_{\text{ungerade}}\,dy = 0$$

$$\langle p_1, p_1\rangle = \int_{-1}^{1} \frac{y^2}{\sqrt{1-y^2}}\,dy = \int_0^{\pi} \cos^2 x\,dx = \int_0^{\pi} \frac{1+\cos 2x}{2}\,dx = \frac{\pi}{2}$$

$$\underline{\bar{p}_1 = \sqrt{\frac{2}{\pi}}\,x}$$

$$\langle \bar{p}_0, p_2\rangle = \int_{-1}^{1} \frac{y^2\frac{1}{\sqrt{\pi}}}{\sqrt{1-y^2}}\,dy = \frac{\pi}{2}\frac{1}{\sqrt{\pi}}$$

$$\langle p_1, p_2\rangle = \int_{-1}^{1} \frac{y^3}{\sqrt{1-y^2}}\,dy = 0$$

$$\tilde{p}_2 = p_2 - \frac{\sqrt{\pi}}{2}\bar{p}_0 = x^2 - \frac{1}{2}$$

$$\langle \tilde{p}_2, \tilde{p}_2\rangle = \int_{-1}^{1} \frac{(y^2-\frac{1}{2})^2}{\sqrt{1-y^2}}\,dy = \int_{-1}^{1} \frac{y^4+\frac{1}{4}-y^2}{\sqrt{1-y^2}}\,dy = \int_0^{\pi}\left(\cos^4 x + \frac{1}{4} - \cos^2 x\right)dx = \frac{3\pi}{8} + \frac{\pi}{4} - \frac{\pi}{2} = \frac{\pi}{8}$$

$$\int_0^{\pi} \cos^4 x\,dx = \begin{bmatrix} \cos^3 x \to -3\cos^2 x \sin x \\ \cos x \leftarrow \sin x \end{bmatrix} = \left[\cos^3 x \sin x\right]_0^{\pi} + 3\int_0^{\pi} \cos^2 x \sin^2 x\,dx = 3\int_0^{\pi}\left(-\cos^4 x + \cos^2 x\right)dx$$

$$\int_0^{4} \cos^4 x\,dx = \frac{3}{4}\int_0^{\pi} \cos^2 x\,dx = \frac{3\pi}{8}$$

$$\underline{\bar{p}_2 = \left(x^2 - \frac{1}{2}\right)2\sqrt{\frac{2}{\pi}}}$$

$$\langle \bar{p}_0, p_3\rangle = \int_{-1}^{1} \frac{y^3\frac{1}{\sqrt{\pi}}}{\sqrt{1-y^2}}\,dy = 0$$

$$\langle \bar{p}_1, p_3\rangle = \int_{-1}^{1} \frac{\sqrt{\frac{2}{\pi}}\,y^4}{\sqrt{1-y^2}}\,dy = \sqrt{\frac{2}{\pi}}\,\frac{3\pi}{8} = \frac{3}{8}\sqrt{2\pi}$$

$$\langle \bar{p}_2, p_3\rangle = \int_{-1}^{1} \frac{y^3\, 2\sqrt{\frac{2}{\pi}}\left(y^2-\frac{1}{2}\right)}{\sqrt{1-y^2}} = 0$$

$$\tilde{p}_3 = x^3 + \frac{3}{8}\sqrt{2\pi}\,\frac{\sqrt{2}}{\sqrt{\pi}}\,x = x^3 + \frac{3x}{4}$$

$$\langle \tilde{p}_3, \tilde{p}_3 \rangle = \int_{-1}^{1} \frac{\left(y^3 + \frac{3y}{4}\right)^2}{\sqrt{1-y^2}}\,dy = \int_0^{\pi} \left(\cos^6 x + \frac{3}{2}\cos^4 x + \frac{9}{16}\cos^2 x\right) dx = \frac{15}{48}\pi + \frac{9}{16}\pi + \frac{9}{32}\pi$$

$$\int_0^{\pi} \cos^6 x\,dx = \begin{bmatrix} \cos^5 x \to -5\cos^4 x \sin x \\ \cos x \leftarrow \sin x \end{bmatrix} = -5\int_0^{\pi} \cos^4 x \sin^2 x\,dx = -5\int_0^{\pi}\left(\cos^4 x - \cos^6 x\right) dx$$

$$\int_0^{\pi} \cos^6 x\,dx = \frac{5}{6}\int_0^{\pi} \cos^4 x\,dx = \frac{15}{48}\pi = \frac{5}{16}\pi$$

$$\underline{\bar{p}_3 = \left(4x^3 + 3x\right)\frac{1}{4}\sqrt{\frac{32}{37\pi}} = \left(4x^3+3x\right)\sqrt{\frac{2}{37\pi}}}$$

$$\langle \bar{p}_0, p_4 \rangle = \int_{-1}^{1} \frac{y^4 \frac{1}{\sqrt{\pi}}}{\sqrt{1-y^2}}\,dy = \frac{1}{\sqrt{\pi}}\int_0^{\pi} \cos^4 x = \frac{3}{8}\sqrt{\pi}$$

$$\langle \bar{p}_1, p_4 \rangle = \int_{-1}^{1} \frac{y^5 \sqrt{\frac{2}{\pi}}}{\sqrt{1-y^2}}\,dy = 0$$

$$\langle \bar{p}_2, p_4 \rangle = \int_{-1}^{1} \frac{y^4\left(y^2 - \frac{1}{2}\right)2\sqrt{\frac{2}{\pi}}}{\sqrt{1-y^2}}\,dy = \int_0^{\pi} 2\sqrt{\frac{2}{\pi}}\left(\cos^6 x - \frac{1}{2}\cos^4 x\right) dx = \frac{15}{24}\sqrt{2\pi} - \frac{3}{8}\sqrt{2\pi} = \frac{\sqrt{2\pi}}{4}$$

$$\langle \bar{p}_3, p_4 \rangle = \int_{-1}^{1} \frac{y^4 \sqrt{\frac{2}{37\pi}}\left(4y^3 + 3y\right)}{\sqrt{1-y^2}}\,dy = 0$$

$$\tilde{p}_4 = x^4 - \frac{3}{8} - \left(x^2 - \frac{1}{2}\right) = x^4 - x^2 + \frac{1}{8}$$

$$\langle \tilde{p}_4, \tilde{p}_4 \rangle = \int_{-1}^{1} \frac{\left(y^4 - y^2 + \frac{1}{8}\right)^2}{\sqrt{1-y^2}}\,dy = \int_0^{\pi}\left(\cos^8 x - 2\cos^6 x + \frac{5}{4}\cos^4 x - \frac{1}{4}\cos^2 x + \frac{1}{64}\right) dx$$

$$\int_0^{\pi} \cos^8 x\,dx = \begin{bmatrix} \cos^7 x \to -7\sin x \cos^6 x \\ \cos x \leftarrow \sin x \end{bmatrix} = 7\int_0^{\pi} \cos^6 x\left(1 - \cos^2 x\right) dx$$

$$\int_0^{\pi} \cos^8 x\,dx = \frac{7}{8}\int_0^{\pi} \cos^6 x\,dx = \frac{7}{8}\cdot\frac{5}{16}\pi = \frac{35}{128}\pi$$

$$\bar{p}_4 = \left(y^4 - y^2 + \frac{1}{8}\right)\left(\frac{35}{128}\pi - 2\frac{5}{16}\pi + \frac{5}{4}\frac{3}{8}\pi - \frac{1}{4}\frac{\pi}{2} + \frac{\pi}{64}\right)^{-\frac{1}{2}}$$

$$\bar{p}_4 = \left(y^4 - y^2 + \frac{1}{8}\right)\left(\frac{\pi}{128}\right)^{-\frac{1}{2}}$$

2) $p_0 = 1$ $\qquad$ $Lp_0 = 0$

$p_1 = x$ $\qquad$ $Lp_1 = -1 = -p_0$

$p_2 = 2x^2 - 1$ $\qquad$ $Lp_2 = -4x = -4p_1$

$p_3 = 4x^3 - 3x$ $\qquad$ $Lp_3 = -12x^2 + 3 = -6p_2 - 3p_0$

$p_4 = 8x^4 - 8x^2 + 1$ $\qquad$ $Lp_4 = -32x^3 + 16x = -8p_3 - 8p_1$

$$
{}^{B}L^{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & -6 & -3 & 0 \\ 0 & 0 & 0 & -8 & -8 \end{pmatrix}
$$

Auf 3)

$1 \Rightarrow 3 \Rightarrow 2 \Rightarrow 4 \Rightarrow 1$

$1 \Rightarrow 3$:

$\forall \varepsilon > 0 \ \exists \delta > 0 : \|h\| < \delta \Rightarrow \|Lh\| < \varepsilon$

$y = x + h$

$\|Ly - Lx\| = \|Lx + Lh - Lx\| = \|Lh\|$

$\|x - y\| = h$

$\Rightarrow L$ stetig in x

$3 \Rightarrow 2$:

$\forall \varepsilon > 0 \ \forall x \ \exists \delta > 0 : \|x - y\| < \delta \Rightarrow \|Lx - Ly\| < \delta$

$\exists \delta > 0 : \|x\| < \delta \Rightarrow \|Lx\| < 1$

$x(y) = \frac{\delta}{2} \frac{y}{\|y\|} \quad \|x(y)\| = \frac{\delta}{2} < \delta \Rightarrow \|L(x(y))\| < 1$

$\|L(x(y))\| = \frac{\delta}{2\|y\|} \|Ly\| < 1$

$\Rightarrow \sup\limits_{y \in V} \frac{\|Ly\|}{\|y\|} \leq \frac{2}{\delta} < \infty$

$2 \Rightarrow 4$:

$\|L(x-y)\| < \|L\| \, \|x-y\|$

$\delta = \frac{\varepsilon}{\|L\|} \Rightarrow \|x-y\| < \delta \Rightarrow \|L(x-y)\| < \varepsilon$

$\Rightarrow \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in B_R^V(0) : \|x-y\| < \delta \Rightarrow \|L(x-y)\| < \varepsilon$

$4 \Rightarrow 1$:

trivial

Aufg 4)

a) $(x,y) \neq 0$: Komposition stetiger Funktionen stetig

$(x,y) = 0$: $|f(x,y) - f(0,0)| = \left| \frac{y^3 - x^2 y}{x^2+y^2} \right| \leq |y| \frac{x^2+y^2}{x^2+y^2} = |y| \Rightarrow \lim\limits_{(x,y)\to(0,0)} |f(x,y) - f(0,0)| = 0$

f stetig in $\mathbb{R}^2$

b) $(x,y) \neq 0$: $\frac{\partial f}{\partial x}(x,y) = \frac{-2xy(x^2+y^2) - 2x(y^3 - x^2 y)}{(x^2+y^2)^2} = \frac{-4xy^3}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y}(x,y) = \frac{3y^2(x^2+y^2) - x^2(x^2+y^2) - 2y(y^3 - x^2 y)}{(x^2+y^2)^2} = \frac{y^4 + 4x^2y^2 - x^4}{(x^2+y^2)^2}$

$(x,y) = (0,0)$: $\frac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \frac{f(h,0) - 0}{h} = 0$

$\frac{\partial f}{\partial y}(0,0) = \lim\limits_{h\to 0} \frac{f(0,h) - 0}{h} = 1$

c) $(x,y) \to \left(\frac{1}{k}, \frac{1}{k}\right) \Rightarrow \lim\limits_{k\to 0} \frac{\partial f}{\partial x}\left(\frac{1}{k}, \frac{1}{k}\right) = -1 \neq 0$; $\frac{\partial f}{\partial x}$ unstetig in 0

$(x,y) \to \left(\frac{1}{k}, 0\right) \Rightarrow \lim\limits_{k\to 0} \frac{\partial f}{\partial y}\left(\frac{1}{k}, 0\right) = -1 \neq 1$; $\frac{\partial f}{\partial y}$ unstetig in 0

d) $D_v f(0,0) = \lim\limits_{\lambda\to 0} \frac{f(\lambda \vec{v}) - f(0,0)}{\lambda} = \lim\limits_{\lambda\to 0} \frac{\lambda^3}{\lambda^3} \frac{(v_y^3 - v_x^2 v_y)}{(v_x^2 + v_y^2)} = \frac{v_y^3 - v_x^2 v_y}{v_x^2 + v_y^2}$

$D_v f(0,0) \overset{!}{=} \nabla f(0,0) \cdot \vec{v} = v_y \Rightarrow \frac{v_y^2 - v_x^2}{v_x^2 + v_y^2} v_y = v_y \qquad v_y = 0$ ✓

$v_y^2 - v_x^2 = v_x^2 + v_y^2$

$v_x^2 = 0$

$\Rightarrow D_v f(0,0) = \nabla f(0,0) \cdot \vec{v} \Rightarrow v_y = 0 \vee v_x = 0$