Zu 1)

$\oplus: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ ✓

$\odot: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ ✓

V1: $x \cdot (y \cdot z) = x \cdot y \cdot z = (x \cdot y) \cdot z$ Assoziativgesetz

V2: $x \cdot 1 = x = 1 \cdot x$ Existenz eines neutralen Elements

V3: $x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x$ Existenz eines inversen Elements

V4: $x \cdot y = y \cdot x$ Kommutativgesetz

S1: $(x \cdot y)^\alpha = x^\alpha \cdot y^\alpha$

S2: $x^{\alpha + \beta} = x^\alpha \cdot x^\beta$

S3: $(x)^{\alpha\beta} = (x^\alpha)^\beta$

S4: $x^1 = x$

Vektorraum

Eigenschaften $\forall u,v,w \in V, \forall \alpha,\beta \in K$:

V1: $u \oplus (v \oplus w) = (u \oplus v) \oplus w$

V2: $\exists 0_V \in V: u \oplus 0_V = 0_V \oplus u = u$

V3: $\exists -v \in V: v \oplus (-v) = (-v) \oplus v = 0_V$

V4: $v \oplus u = u \oplus v$

S1: $\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$

S2: $(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v)$

S3: $(\alpha \cdot \beta) \odot v = \alpha \odot (\beta \odot v)$

S4: $\exists 1_k \in K: 1_k \odot v = v$

Zu 2. man muss zeigen $\oplus, \odot$ abgeschlossen in $V \wedge 0_V \in V$

a) nein: $\odot: \mathbb{C} \times V \not\to V$ $\quad i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}}_{\notin V}$

b) ja: 1) $\oplus$: $\begin{pmatrix} 0 \\ a_1 \\ a_2 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ a_1+b_1 \\ a_2+b_2 \end{pmatrix}$ $\quad$ 2) $\odot$: $\alpha \begin{pmatrix} 0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha a_1 \\ \alpha a_2 \end{pmatrix}$ $\quad$ 3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$

c) nein $\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\in V} \oplus \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\in V} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}_{\notin V}$

d) ja: 1) $\underbrace{\begin{pmatrix} 2a_1 \\ -a_1 \\ a_2 \end{pmatrix}}_{\in V} + \underbrace{\begin{pmatrix} 2b_1 \\ -b_1 \\ b_2 \end{pmatrix}}_{\in V} = \underbrace{\begin{pmatrix} 2(b_1+a_1) \\ -(a_1+b_1) \\ a_2+b_2 \end{pmatrix}}_{\in V}$ $\quad$ 2) $\alpha \begin{pmatrix} 2a_1 \\ -a_1 \\ a_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2\alpha a_1 \\ -\alpha a_1 \\ -\alpha a_2 \end{pmatrix}}_{\in V}$ $\quad$ 3) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$

e) nein: $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin V$

f) nein: $\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} a_2 \\ a_2 \\ a_1 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1+a_2 \\ a_1+a_2 \\ a_1+a_2 \end{pmatrix}}_{\notin V}$

$a_1 \neq a_2$

g) ja: $\begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_1 \\ b_2 \end{pmatrix} = \underbrace{\begin{pmatrix} a_1+b_1 \\ a_1+b_1 \\ a_2+b_2 \end{pmatrix}}_{\in V}$, $\alpha \begin{pmatrix} a_1 \\ a_1 \\ a_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha a_1 \\ \alpha a_1 \\ \alpha a_2 \end{pmatrix}}_{\in V}$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$

h) nein $\begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} + \begin{pmatrix} a_2 \\ a_1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1+a_2 \\ a_1+a_2 \\ 0 \end{pmatrix} \notin V$ $\quad a_1 \neq a_2$

Au 3)

1.

$\oplus : V \times V \to V \qquad p(x)+q(x)=r(x) \checkmark \quad \sum_{j=0}^{n} a_j x^j + \sum_{k=0}^{m} b_k x^k = \sum_{l=0}^{\max(n,m)} (a_l+b_l) x^l$

$\odot : \mathbb{R} \times V \to V \qquad \alpha \cdot p(x) = q(x) \checkmark \quad q(x) = (\alpha a_0) + (\alpha a_1) x + \ldots + (\alpha a_n) x^n$

V1: $p(x) + (q(x) + r(x)) = (p(x) + q(x)) + r(x)$

V2: $p(x) + 0 = p(x) = 0 + p(x) \qquad 0_V = 0$

V3: $p(x) + (-p(x)) = 0 = (-p(x)) + p(x) \qquad (-p(x)) = -a_0 - a_1 x + \ldots - a_n x^n$

V4: $p(x) + q(x) = q(x) + p(x)$

S1: $\alpha(p(x) + q(x)) = \alpha p(x) + \alpha q(x)$

S2: $(\alpha + \beta) p(x) = \alpha p(x) + \beta p(x)$

S3: $(\alpha \cdot \beta) p(x) = \alpha(\beta p(x))$

S4: $1 \cdot p(x) = p(x)$

2. $(Dp)(x) = \left(\sum_{j=0}^{n} a_j x^j\right)' = \sum_{j=0}^{n} a_j (x^j)' = \sum_{j=0}^{n-1} a_j j x^{j-1}$

$D(\alpha p + q)(x) = D\left(\alpha \sum a_j x^j + \sum b_j x^j\right) = D \sum^{\max\{n,m\}} (\alpha a_j + b_j) x^j \qquad a_j = 0,\ b_k = 0 \quad j > n,\ k > m$

$= \sum (\alpha a_j + b_j) j x^{j-1} = \sum \alpha a_j j x^{j-1} + \sum b_j j x^{j-1}$

$= \alpha D\left(\sum a_j x^j\right) + D\left(\sum b_j x^j\right)$

$J(\alpha p + q)(x) = \int_0^x (\alpha p(t) + q(t))\, dt = \alpha \int_0^x p(t)\, dt + \int_0^x q(t)\, dt = \alpha J(p)(x) + \beta J(q)(x)$

3. $(DJ(p))(x) = D \int_0^x p(t)\, dt = D[P(x) - P(0)] = p(x) \qquad P'(x) = p(x)$

$(JD(p))(x) = J \sum_{j=0}^{n} j a_j x^{j-1} = J \sum_{j=1}^{n} j a_j x^{j-1} = \sum_{j=1}^{n} a_j J(j x^{j-1}) = \sum_{j=1}^{n} a_j x^j$

$JD\left(\sum_{j=0}^{n} a_j x^j\right) = \sum_{j=1}^{n} a_j x^j$

4. nein, weil $JD \neq DJ$

An 4:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_d \end{pmatrix} \begin{pmatrix} b_1 & b_2 & \dots & b_d \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_d \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_d \\ \vdots & \vdots & & \vdots \\ a_d b_1 & a_d b_2 & \dots & a_d b_d \end{pmatrix}$$

$$Cx = \begin{pmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_d \\ a_2 b_1 & \ddots & & \vdots \\ \vdots & & & \\ a_d b_1 & \dots & & a_d b_d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} \sum_j a_1 b_j x_j \\ \sum_j a_2 b_j x_j \\ \vdots \\ \sum_j a_d b_j x_j \end{pmatrix} = \begin{pmatrix} a_1 \sum b_j x_j \\ a_2 \sum b_j x_j \\ \vdots \\ a_d \sum b_j x_j \end{pmatrix}$$

$$\langle b, x \rangle a = \sum b_j x_j \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} a_1 \sum b_j x_j \\ a_2 \sum b_j x_j \\ \vdots \\ a_d \sum b_j x_j \end{pmatrix}$$

Au 5)

nein $x^d - x^d = 0$ ← P. von Grad 0

P. von Grad d P. von Grad d

Aufg.

1) Beweis von Definition: $(\exists a, b: a\vec{x} + b\vec{y} = 0 \Rightarrow a, b = 0)$

$\exists a, b \quad \forall x: a \sin x + b \cos x = 0$

$x_1 = \frac{\pi}{4}: \quad a \sin\frac{\pi}{4} + b \cos\frac{\pi}{4} = 0 \Rightarrow a = -b \Rightarrow b = -a$

$x_2 = 0: \quad 0 = a \sin 0 + b \cos 0 \Rightarrow b = 0 \Rightarrow a = 0$

2) Beweis von Definition

$\exists a, b \quad \forall x: a e^x + b \cos x = 0$

$x_1 = 0: \quad a e^0 + b \cos 0 = 0 \Rightarrow a = -b \Rightarrow b = -a$

$x_2 = \frac{\pi}{2}: \quad a e^{\frac{\pi}{2}} + b \cos\frac{\pi}{2} = 0 \Rightarrow a = 0 \Rightarrow b = 0$

3) Beweis von Definition

$\exists a, b, c \quad , a, b, c \in \mathbb{R} \ \forall x: a e^{ix} + b \sin x + c \cos x = 0$

$x = 0: \quad a e^0 + b \sin 0 + c \cos 0 = 0 \Rightarrow a = -c$

$x = \frac{\pi}{2}: \quad a e^{i\frac{\pi}{2}} + b \sin\frac{\pi}{2} + c \cos\frac{\pi}{2} = 0 \Rightarrow ai + b = 0 \Rightarrow a = 0, b = 0 \Rightarrow c = 0$

mit $\mathbb{R}$ lin. unabhängig

mit $\mathbb{C}$: $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

$\Rightarrow i \sin x + \cos x = e^{ix} \Rightarrow i \cdot \sin x + 1 \cdot \cos x - e^{ix} = 0$

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$

du7)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 31 \\ 0 & 1 & -1 & 6 \\ 1 & 0 & 1 & 7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 31 \\ 0 & 1 & -1 & 6 \\ 0 & -2 & -2 & -24 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 31 \\ 0 & 1 & -1 & 6 \\ 0 & -1 & -1 & -12 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 31 \\ 0 & 1 & -1 & 6 \\ 0 & 0 & -2 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$x_1 = 4$

$x_2 = 9$

$x_3 = 3$

Au 8.

1) $\mathbb{R}$ mit std. Verknüpfungen Körper $\Rightarrow$ V1 $\to$ V4, S1, S2, S3 sind wahr

S4: $1 \in \mathbb{Q}$ $\quad 1 \cdot x = x$ ✓

$\mathbb{R}$ über $\mathbb{Q}$ Vektorraum ✓

2) nein $\quad \odot : \mathbb{R} \otimes \mathbb{Q} \not\to \mathbb{Q}$

z.B. $\quad \pi \cdot 1 = \pi \notin \mathbb{Q}$