Aufg. 52)

a)

$$Df(x,y) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{x} & \frac{1}{y} \\ -2x\sin(x^2+y) & -\sin(x^2+y) \\ e^x & 0 \end{pmatrix}$$

$$Dg(x,y,z) = \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} = \begin{pmatrix} e^x & z & y+\frac{1}{z} \end{pmatrix}$$

b)

$$(Dg)(f(x,y)) = \begin{pmatrix} xy & e^x & \cos(x^2+y)+e^{-x} \end{pmatrix}$$

$$D(g\circ f) = (Dg)(f(x,y))\,Df(x,y) = \begin{pmatrix} y-2xe^x\sin(x^2+y)+1+e^x\cos(x^2+y) & x-e^x\sin(x^2+y) \end{pmatrix}$$

c)

$$(g\circ f)(x,y) = xy + e^x\cos(x^2+y) + x = h(x,y)$$

$$Dh(x,y) = \begin{pmatrix} y+1+e^x\cos(x^2+y)-2xe^x\sin(x^2+y) & x-e^x\sin(x^2+y) \end{pmatrix}$$

Auf 53)

a)
$$Df(x,y)=\begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}=\begin{pmatrix}2x & 0\\ 0 & 2y\end{pmatrix}$$

$$Dg(x,y)=\begin{pmatrix}\frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y}\\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y}\end{pmatrix}=\begin{pmatrix}y\cos(xy) & x\cos(xy)\\ e^{x+y} & e^{x+y}\end{pmatrix}$$

$$Dh(x,y)=\begin{pmatrix}\frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y}\\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y}\end{pmatrix}=\begin{pmatrix}e^{x}\cos(y) & -e^{x}\sin(y)\\ \cosh(x) & 0\end{pmatrix}$$

b)
$$(Dg)(f(x,y))=\begin{pmatrix}y^2\cos(x^2y^2) & x^2\cos(x^2y^2)\\ e^{x^2+y^2} & e^{x^2+y^2}\end{pmatrix}$$

$$D(g(f(x,y)))=(Dg)(f(x,y))\,Df(x,y)=\begin{pmatrix}2xy^2\cos(x^2y^2) & 2yx^2\cos(x^2y^2)\\ 2x\,e^{x^2+y^2} & 2y\,e^{x^2+y^2}\end{pmatrix}$$

$$(Dh)(g(x,y))=\begin{pmatrix}e^{\sin(xy)}\cos(e^{x+y}) & -e^{\sin xy}\sin(e^{x+y})\\ \cosh(\sin(xy)) & 0\end{pmatrix}$$

$$D(h(g(x,y)))=(Dh)(g(x,y))\,Dg(x,y)=\begin{pmatrix}e^{\sin(xy)}\left(\cos(e^{x+y})\,y\cos(xy)-e^{x+y}\sin(e^{x+y})\right) & \times\\ \cosh(\sin(xy))\,e^{x+y} & \oplus\end{pmatrix}$$

$$\times = e^{\sin(xy)}\left(\cos(e^{x+y})\,x\cos(xy)-\sin(e^{x+y})\,e^{x+y}\right)$$
$$\oplus = x\cos(xy)\cosh(\sin(xy))$$

c)
$$(g\circ f)(x,y)=\left(\sin(x^2y^2),\,e^{x^2+y^2}\right)=i(x,y)$$

$$Di(x,y)=\begin{pmatrix}2xy^2\cos(x^2y^2) & 2yx^2\cos(x^2y^2)\\ 2x\,e^{x^2+y^2} & 2y\,e^{x^2+y^2}\end{pmatrix}$$

Auf 54)

Vektorraum:

$A, B, C \in \mathcal{L}(V, W)$, $x \in V$ $Ax, Bx, Cx \in W$

V1) $((A+B)+C)x = Ax + Bx + Cx = Ax + (B+C)x = (A+(B+C))x$ ✓

V2) $(A + 0 \cdot I)x = Ax + 0 = 0 + Ax = (0I + A)x$ ✓

V3) $(A + (-A))x = Ax - Ax = 0 = 0 \cdot x$ ✓

V4) $(A+B)x = Ax + Bx = Bx + Ax = (B+A)x$

S1) $\alpha(A+B)x = \alpha Ax + \alpha Bx = (\alpha A + \alpha B)x$ ✓

S2) $(\alpha + \beta)Ax = \alpha Ax + \beta Ax = (\alpha A + \beta A)x$ ✓

S3) $(\alpha \cdot \beta)Ax = \alpha(\beta Ax) = \alpha \beta Ax$

S4) $1 \cdot Ax = Ax$

Norm:

1) $\|x\| = 0 \iff x = 0$: $\sup\limits_{u \in V, u \neq 0} \frac{\|Lu\|_W}{\|u\|} = 0 \iff Lu = 0 \Rightarrow L = 0$

2) $\|\lambda x\| = |\lambda| \cdot \|x\|$: $\sup\limits_{u \in V, u \neq 0} \frac{\|\lambda L u\|_W}{\|u\|} = \sup\limits_{u \in V, u \neq 0} \frac{|\lambda| \|Lu\|_W}{\|u\|} = |\lambda| \sup\limits_{u \in V, u \neq 0} \frac{\|Lu\|_W}{\|u\|}$

3) $\|x+y\| \leq \|x\| + \|y\|$:

$$\sup_{u \in V, u \neq 0} \frac{\|(A+B)u\|_W}{\|u\|} = \sup \frac{\|Au + Bu\|_W}{\|u\|} \leq \sup \frac{\|Au\|_W + \|Bu\|_W}{\|u\|} \leq$$

$$\leq \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|Au\|_W}{\|u\|} + \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|Bu\|_W}{\|u\|}$$

$$h(g(x,y)) = \begin{pmatrix} e^{\sin(xy)} \cos(e^{x+y}) & \sinh(\sin(xy)) \end{pmatrix} = j(x,y)$$

$$Dj(x,y) = \begin{pmatrix} y\cos(xy)\, e^{\sin xy} \cos(e^{x+y}) - \sin(e^{x+y})\, e^{x+y} e^{\sin(xy)} & \text{⧖} \\ y\cos(xy)\cosh(\sin(xy)) & x\cos(xy)\cosh(\sin(xy)) \end{pmatrix}$$

$$\text{⧖} = x\cos(xy)\, e^{\sin xy} \cos(e^{x+y}) - e^{x+y} e^{\sin(xy)} \sin(e^{x+y})$$

Auf 55)

$\|\cdot\|_\infty$:

Axioms

1) $\|y\|_\infty = 0 = \max\limits_{1\le j\le n} \|y_j\|_{W_j} \Rightarrow \|y_j\|_{W_j} = 0 \Rightarrow y_j = 0 \Rightarrow y = 0$

$y = 0 \Rightarrow y_j = 0 \Rightarrow \|y_j\|_{W_j} = 0 \Rightarrow \|y\|_\infty = 0$

2) $\|\alpha y\|_\infty = \max\limits_{1\le j\le n} \|\alpha y_j\|_{W_j} = \max\limits_{1\le j\le n} |\alpha| \|y_j\|_{W_j} = |\alpha| \max\limits_{1\le j\le n} \|y_j\|_{W_j}$

3) $\|x+y\|_\infty = \max\limits_{1\le j\le n} \|x_j+y_j\|_{W_j} \le \max\limits_{1\le j\le n} \left(\|x_j\|_{W_j} + \|y_j\|_{W_j}\right) \le \max\limits_{1\le j\le n} \|x_j\|_{W_j} + \max\limits_{1\le j\le n} \|y_j\|_{W_j}$

$\|\cdot\|_p$:

Axioms

1) $\|y\|_p = 0 \Rightarrow \|y\|_p^p = 0 \Rightarrow \sum\limits_{j=1}^n \|y_j\|_{W_j} = 0 \Rightarrow \|y_j\|_{W_j} = 0 \Rightarrow y_j = 0 \Rightarrow y = 0$

$y = 0 \Rightarrow y_j = 0 \Rightarrow \|y_j\|_{W_j} = 0 \Rightarrow \|y\|_p = 0$

2) $\|\alpha y\|_p = \left(\sum\limits_{j=1}^n \|\alpha y_j\|_{W_j}^p\right)^{\frac{1}{p}} = \left(\sum\limits_{j=1}^n |\alpha|^p \|y_j\|_{W_j}^p\right)^{\frac{1}{p}} = |\alpha| \left(\sum\limits_{j=1}^n \|y_j\|_{W_j}^p\right)^{\frac{1}{p}}$

3) $\|x+y\|_p^p = \left(\sum\limits_{j=1}^n \|x_j+y_j\|_{W_j}^p\right) \le \underbrace{\left(\sum\limits_{j=1}^n \left(\|\overline{x}_j\|_{W_j}^p (1-\lambda) + \|\overline{y}_j\|_{W_j}^p \lambda\right)\right)}_{\le 1} \left(\|x\|_p + \|y\|_p\right)^p$

$\lambda = \frac{\|y\|_p}{\|x\|_p + \|y\|_p}$ $\quad \overline{x} = \frac{x}{\|x\|_p}$ $\quad \overline{y} = \frac{y}{\|y\|_p}$

$\|x_j + y_j\|_{W_j}^p = \left(\|x\|_p + \|y\|_p\right)^p \|(1-\lambda)\overline{x}_j + \lambda \overline{y}_j\|_{W_j}^p \underset{\uparrow}{\le} \left(\|x\|_p + \|y\|_p\right)^p \left[|1-\lambda| \|\overline{x}_j\|_{W_j}^p + |\lambda| \|\overline{y}_j\|_{W_j}^p\right]$

weil $\|\cdot\|_{W_j}^p$ ist konvex: x^p konvex + wachsend, $\|\ \|_{W_j}$ konvex

$\|y\|_\infty = \max\limits_{1\le j\le n} \left(\|y_j\|_{W_j}^p\right)^{\frac{1}{p}} \le \left(\sum\limits_{j=1}^n \|y_j\|_{W_j}^p\right)^{\frac{1}{p}} = \|y\|_p \le \left(\sum\limits_{k=1}^n \max\limits_{1\le j\le n} \|y_j\|_{W_j}^p\right)^{\frac{1}{p}} = n^{\frac{1}{p}} \underbrace{\left(\max\limits_{1\le j\le n} \|y_j\|_{W_j}^p\right)^{\frac{1}{p}}}_{\|y\|_\infty}$

$\|\ \|_{W_j}$ convex: $\|(1-\lambda)x + \lambda y\|_{W_j} \le \|(1-\lambda)x\|_{W_j} + \|\lambda y\|_{W_j} = |1-\lambda| \|x\|_{W_j} + |\lambda| \|y\|_{W_j}$

x^p convex:

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Beweis: $3 \Rightarrow 1 \Rightarrow 4 \Rightarrow 2 \Rightarrow 3$

$3 \Rightarrow 1$:

$\forall \varepsilon > 0 \ \exists \delta > 0 : \|x\| < \delta \Rightarrow \|Lx\| < \varepsilon$

$y = x + h$

$\|Lx - L(x+h)\| = \|Lx - Lx - Lh\| = \|Lh\|$

$x - y = x - x - h = h$

$\Rightarrow \forall \varepsilon > 0 \ \forall x \ \exists \delta > 0 : \|x-y\| = \|h\| < \delta \Rightarrow \|L(x-y)\| < \varepsilon$

$1 \Rightarrow 4$:

$\forall \varepsilon > 0 \ \forall x \ \exists \delta > 0 : \|x-y\| < \delta \Rightarrow \|Lx - Ly\| < \delta$

$\exists \delta > 0 : \|x\| < \delta \Rightarrow \|Lx\| < 1$

$x(y) = \frac{\delta}{2} \frac{y}{\|y\|} \qquad \|x(y)\| = \frac{\delta}{2} < \delta \Rightarrow \|L(x(y))\| < 1$

$\|L(x(y))\| = \frac{\delta}{2\|y\|} \|Ly\| < 1$

$\sup_{y \neq 0} \frac{\|Ly\|}{\|y\|} \leq \frac{2}{\delta} < \infty$

$4 \Rightarrow 2$

$\|L(x-y)\| < \|L\| \, \|x-y\|$

$\delta = \frac{\varepsilon}{\|L\|} \Rightarrow \|x-y\| < \delta \Rightarrow \|L(x-y)\| < \varepsilon$

$2 \Rightarrow 3$

trivial

Auf 57)

$\Rightarrow$: $\forall \varepsilon>0\ \exists \delta>0: \|x-x_0\|<\delta \Rightarrow \|g(x)-g(x_0)\|_{\mathcal{L}}<\varepsilon$

wir nehmen $h_i = (\underset{1.}{0}, \dots 0, \underset{i.}{1}, 0 \dots, \underset{n.}{0})$

$\|g(x)h_i - g(x_0)h_i\| = \|u_i(x)-u_i(x_0)\| \leq \|g(x)-g(x_0)\|_{\mathcal{L}} < \varepsilon$

$\forall \varepsilon>0\ \exists \delta>0: \|x-x_0\|<\delta \Rightarrow \|u_i(x)-u_i(x_0)\|<\varepsilon$

$\Leftarrow$: $\forall \varepsilon>0\ \exists \delta>0: \|x-x_0\|<\delta \Rightarrow \|u_j(x)-u_j(x_0)\|<\varepsilon\frac{1}{n}$

$\|g(x)-g(y)\| = \sup_{\|h\|_\infty=1} \|(g(x)-g(x_0))[h]\|_W \leq \sum_{j=1}^{n} \|u_j(x)-u_j(x_0)\|_W < \varepsilon$