58)

a) 3 Möglichkeiten:

1) $(u,v) \in \mathbb{R}^{2m} \mapsto \langle u, v \rangle$ differenzierbar

Kombination von differenzierbaren Funktionen ist differenzierbar.

2) $h(x+\varepsilon) = h(x) - Dh(x)[\varepsilon] + \|\varepsilon\| \varepsilon_x(\varepsilon)$

$\langle f(x+\varepsilon), g(x+\varepsilon) \rangle - \langle f(x), g(x) \rangle = \langle f(x+\varepsilon) - f(x), g(x+\varepsilon) \rangle + \langle f(x), g(x+\varepsilon) - g(x) \rangle$

$\Rightarrow Dh(x)[\varepsilon] = \langle Df(x)[\varepsilon], g(x) \rangle + \langle f(x), Dg(x)[\varepsilon] \rangle$

3) SSP:

$h(x) = \sum f_i(x) g_i(x) \Rightarrow Dh(x)[\varepsilon] = \nabla h \cdot \varepsilon$

2) $\langle f(x+\varepsilon), g(x+\varepsilon) \rangle = \langle f(x), g(x) \rangle + \|\varepsilon\| \left[\langle \frac{f(x+\varepsilon) - f(x)}{\|\varepsilon\|}, g(x+\varepsilon) \rangle + \langle f(x), \frac{g(x+\varepsilon) - g(x)}{\|\varepsilon\|} \rangle \right]$

$\langle \frac{f(x+\varepsilon) - f(x)}{\|\varepsilon\|}, g(x+\varepsilon) \rangle = \langle \frac{f(x+\varepsilon) - f(x)}{\|\varepsilon\|}, g(x) \rangle + \|\varepsilon\| \langle \frac{f(x+\varepsilon) - f(x)}{\|\varepsilon\|}, Dg(x)[\varepsilon] \rangle + \|\varepsilon\| \varepsilon_x(\varepsilon)$

Auf 59)

f differenzierbar in $x_0 \Rightarrow D_u f(x_0) = \nabla f(x_0) \cdot u$

Wir haben

$g(u) = \nabla f(x_0) \cdot u \qquad \|u\| = 1$

$L = g(u) - \lambda \|u\|^2$

$\Rightarrow \partial_{u_j} L = (\nabla f(x_0))_j - 2\lambda u_j \overset{!}{=} 0$

$$u_j = \frac{(\nabla f(x_0))_j}{2\lambda}$$

$$\|u\|^2 = \frac{1}{4\lambda^2} \|\nabla f(x_0)\|^2 = 1 \Rightarrow 2\lambda = \|\nabla f(x_0)\|$$

$$\Rightarrow u = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$$

$$D_u f(x_0) = \langle \nabla f(x_0), u \rangle \overset{\text{C.S.U.}}{\leq} \|\nabla f(x_0)\| \overbrace{\|u\|}^{1}$$

$$\text{Gl. } {}^{\shortparallel}={}^{\shortparallel} \Leftrightarrow u \parallel \nabla f(x_0)$$

$\Rightarrow u \in S^{n-1} \mapsto D_u f(x_0)$ nimmt Max

in $u_{max} = \dfrac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$ an

Auf 60)

$$\|A\| = \max_{\|x\|=1} \|Ax\| \quad (1)$$

Wir nehmen ONB $(e_1, \dots, e_n)$, wo $e_1 = x$ von (1)

$$\|A\|_{HS}^2 = \sum_{j,k} A_{jk}^2 \overset{(2)}{=} \sum_k \|A e_k\|^2 \geq \|Ax\|^2$$

(2) man muss zeigen, dass Skalarprodukt ist Basisinvariant

$$\|A\|_{HS} \geq \|A\|$$

andere Möglichkeit

$$\|A\|^2 = \max_{\|x\|=1} \|Ax\|^2 = \sum_{j=1}^{m} \left(\sum_{k=1}^{n} A_{jk} x_k \right)^2 \overset{\text{Cauchy Schwarz}}{\leq} \sum_{j=1}^{m} \left[\left(\sum_{k=1}^{n} A_{jk}^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} (x_k)^2 \right)^{\frac{1}{2}} \right]^2 = \sum_{j,k} A_{jk}^2$$

Aufg 61)

a) $\partial_x f(x,y) = y+1$
$\partial_y f(x,y) = x-2$ $\Rightarrow (x,y) = (2,-1)$

$$D_f^2(x,y) = \begin{pmatrix} \partial^2_{xx} f & \partial^2_{xy} f \\ \partial^2_{yx} f & \partial^2_{yy} f \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A(x,y)$$

$\det(A-\lambda) = \lambda^2 - 1 \Rightarrow \lambda = 1, -1 \Rightarrow A$ indefinit $\Rightarrow (2,-1)$ kein lok. Maximum oder Minimum

b) $\partial_x g(x,y) = 6x^2 - 3y \overset{!}{=} 0$
$\partial_y g(x,y) = 6y^2 - 3x \overset{!}{=} 0$ $\Rightarrow x^2 = \frac{y}{2}$, $y^2 = \frac{x}{2}$ $\quad 4y^4 = \frac{y}{2}$, $y \neq 0$: $y^3 = \frac{1}{8}$ $\quad y = 2^{-1}$, $x = 2^{-1}$

$(x,y) = (0,0)$
$(x,y) = (\frac{1}{2}, \frac{1}{2})$

$$D_g^2(x,y) = \begin{pmatrix} 12x & -3 \\ -3 & 12y \end{pmatrix} = B(x,y)$$

$B(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} \Rightarrow \lambda = 3, -3 \Rightarrow B(0,0)$ indefinit $\Rightarrow (0,0)$ kein lok. Maximum oder Minimum

$B(\frac{1}{2}, \frac{1}{2}) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \Rightarrow 0 = \lambda^2 - 12\lambda + 27 \Rightarrow \lambda = 9, 3 \Rightarrow B(\frac{1}{2}, \frac{1}{2})$ positiv definit $\Rightarrow$
$(\frac{1}{2}, \frac{1}{2})$ lok. Minimum $g(\frac{1}{2}, \frac{1}{2}) = -\frac{13}{4}$

c) $\partial_x h(x,y) = 2e^{-x^2-y^2} + (2x+2y+3)(-2x)e^{-x^2-y^2} \overset{!}{=} 0 \Rightarrow 1 = x(2x+2y+3)$
$\partial_y h(x,y) = e^{-x^2-y^2}(-4y^2 - 4xy - 6y + 2) \overset{!}{=} 0 \qquad 1 = y(2x+2y+3)$

$x, y \neq 0 \Rightarrow x = y$, $1 = 4x^2 + 3x \Rightarrow x^2 + \frac{3}{4}x - \frac{1}{4} = 0 \Rightarrow x = -1, \frac{1}{4}$

$$D_h^2(x,y) = \begin{pmatrix} e^{-x^2-y^2}(-4x - 8x - 4y - 6 + 4x^2(2x+2y+3)) & e^{-x^2-y^2}(-4y - 4x + 4xy(2x+2y+3)) \\ e^{-x^2-y^2}(-4y - 4x + 4xy(2x+2y+3)) & e^{-x^2-y^2}(-12y - 4x - 6 + 4y^2(2x+2y+3)) \end{pmatrix}$$

$D_h^2(x,y) = C(x,y)$

$C(-1,-1) = e^{-2}\begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix} \Rightarrow 0 = \lambda^2 - 12\lambda + 20 \Rightarrow \lambda = 10, 2$ positiv definit $\Rightarrow$

$\Rightarrow (-1,-1)$ lok. Minimum $h(-1,-1) = -\frac{1}{e^2}$

$C(\frac{1}{4},\frac{1}{4}) = e^{-\frac{1}{8}}\begin{pmatrix} -9 & -1 \\ -1 & -9 \end{pmatrix} \Rightarrow 0 = \lambda^2 + 18\lambda + 80 \Rightarrow \lambda = -10, -8$ negativ definit $\Rightarrow$

$\Rightarrow (\frac{1}{4},\frac{1}{4})$ lok. Maximum $h(\frac{1}{4},\frac{1}{4}) = 4e^{-\frac{1}{8}}$

Auf 62)

a) von Auf 51)

$D_g(x)[v] = 2\langle x, Lv\rangle$

$D_g(x+u)[v] - D_g(x)[v] = 2\langle x+u, Lv\rangle - 2\langle x, Lv\rangle = 2\langle u, Lv\rangle$

$$\lim_{u\to 0} \frac{\| D_g(x+u)[v] - D_g(x)[v] - 2\langle u, Lv\rangle \|}{|u|} = 0$$

$\Rightarrow D_g(x)[u,v] = 2\langle u, Lv\rangle$

b) $\langle x+h, L^+v\rangle + \langle x+h, Lv\rangle - \langle x, L^+v\rangle - \langle x, Lv\rangle = \langle h, L^+v\rangle + \langle h, Lv\rangle$

$\Rightarrow D_g^2(x)[u,v] = \langle u, L^+v\rangle + \langle u, Lv\rangle$

Auf 63)

a) $f(x) = F(f(x), g(x), h(x))\Big|_{f(x)=g(x)=h(x)=x}$

$$f'(x) = \partial_{f(x)} F \frac{\partial f(x)}{\partial x} + \partial_{g(x)} F \frac{\partial g(x)}{\partial x} + \partial_{h(x)} F \frac{\partial h(x)}{\partial x} = \partial_1 F(x,x,x) + \partial_2 F(x,x,x) + \partial_3 F(x,x,x)$$

b) $F(x_1,x_2,x_3) = \int_{a(x_1)}^{b(x_2)} g(x_3,y)\,dy$

$$\partial_1 F(x_1,x_2,x_3) = -a'(x_1)\, g(x_3, a(x_1))$$
$$\partial_2 F(x_1,x_2,x_3) = b'(x_2)\, g(x_3, b(x_2))$$
$$\partial_3 F(x_1,x_2,x_3) = \int_{a(x_1)}^{b(x_2)} \partial_1 g((x_3), y)\,dy$$

$$f'(x) = -a'(x)\, g(x, a(x)) + b'(x)\, g(x, b(x)) + \int_{a(x)}^{b(x)} \partial_1 g((x), y)\,dy$$