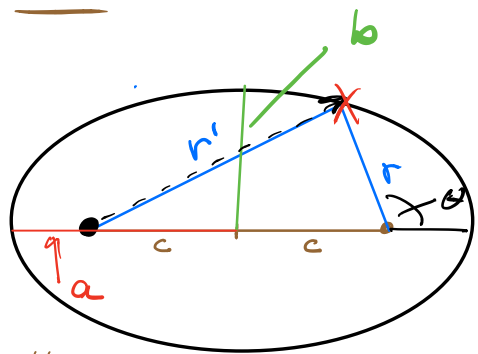
Zusammenfassung d. letzten Vorlesung

Ellipsen



$$\boxed{r(\theta) = \frac{k}{1 + \varepsilon \cos\theta}}$$

$\varepsilon = \frac{c}{a}$, $k = \frac{b^2}{a} = a(1-\varepsilon^2)$

$(c = \sqrt{a^2 - b^2})$

Fläche: $A = \pi a b$

- Brennpunkte

a: große Halbachse

b: kleine Halbachse

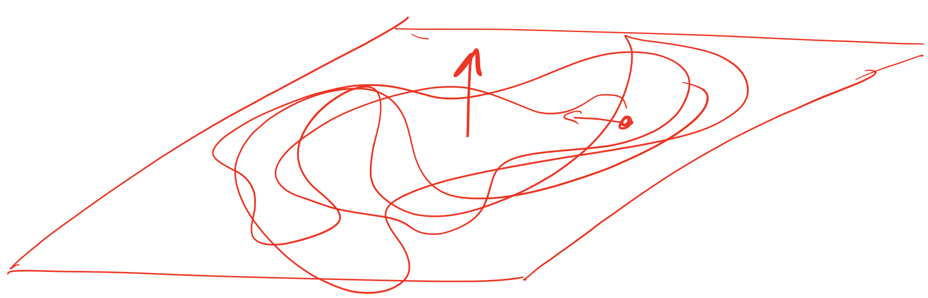
·) Wenn $V(\vec{r}) = V(|\vec{r}|)$ (Zentralpotential)

$\Rightarrow$ $\boxed{\vec{F} = -\frac{\partial V(r)}{\partial r}\vec{e}_r}$ (Radialkraft)

$\Rightarrow$ $\frac{d\vec{L}}{dt} = 0$ mit $\vec{L} = \vec{r} \times \vec{p}$ Drehimpuls

Isotropie des Raumes ergibt Drehimpulserhaltung

$\rightarrow$ Bewegung innerhalb einer Ebene



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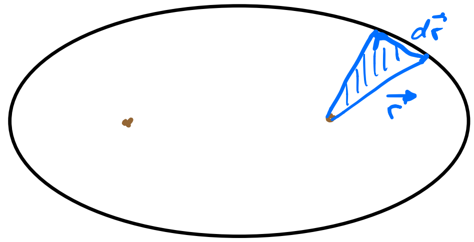


Fläche:

$dA = \frac{1}{2} |\vec{r} \times d\vec{r}|$

$\Rightarrow$ Flächengeschwindigkeit

(überstrichene Fläche pro Zeit)

$$\frac{dA}{dt} = \frac{1}{2} |\vec{r} \times \vec{v}| = \frac{1}{2m} |\vec{L}|$$

$\Longrightarrow$ Drehimpulserhaltung erfordert $\frac{dA}{dt} = \text{const.}$

$\Longrightarrow$ 2. Keplersche Gesetz

$\vec{p} = m\vec{v}$

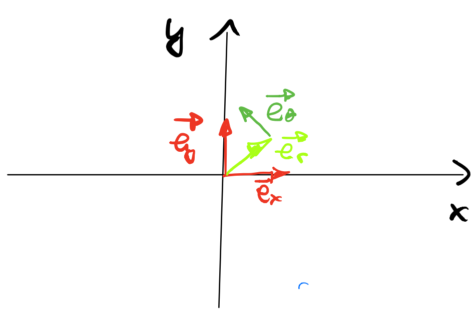
Bahnkurve (x, y) (r, θ)

$$\boxed{\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin\theta \end{aligned}}$$

$$\vec{e}_r = \cos(\theta)\,\vec{e}_x + \sin(\theta)\,\vec{e}_y \qquad \vec{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{e}_\theta = -\sin(\theta)\,\vec{e}_x + \cos(\theta)\,\vec{e}_y \qquad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{e}_r^{\,2} = 1 \qquad \vec{e}_\theta^{\,2} = 1 \qquad \vec{e}_r \cdot \vec{e}_\theta = 0$$



$$\frac{d\vec{e}_r}{dt} = \vec{e}_\theta \frac{d\theta}{dt} \qquad \frac{d\vec{e}_\theta}{dt} = -\vec{e}_r \frac{d\theta}{dt}$$

$$\vec{r}(t) = r(t)\,\vec{e}_r(t) \qquad \left(= r(t)\cos[\theta(t)]\,\vec{e}_x + r(t)\sin[\theta(t)]\,\vec{e}_y \right)$$

$$\vec{v}(t) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \underline{\underline{\frac{dr}{dt}}}\,\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + \frac{d}{dt}\left(r\frac{d\theta}{dt}\vec{e}_\theta\right)$$

$$= \underline{\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]}\vec{e}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\vec{e}_\theta$$

Zentralpotential $\quad V(\vec{r}) = V(r) \qquad r = |\vec{r}|$

$$\longrightarrow \boxed{\vec{F} = -\frac{\partial V(r)}{\partial r}\vec{e}_r}$$

$$\vec{L} = \vec{r} \times \vec{p} \qquad \frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}}{dt}\times\vec{p}}_{=0} + \underbrace{\vec{r}\times\frac{d\vec{p}}{dt}}_{=0 \text{ wenn}}$$

$(\vec{p} = m\vec{v})$

$$\vec{F} = \frac{d\vec{p}}{dt} \sim \vec{r}$$

$\boxed{\text{Wir wissen}}$:

$$\boxed{\vec{F} = m\vec{a} = m\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r}$$

da $\quad r(t) = r(\theta(t)) \qquad r(\theta) = \dfrac{k}{1+\varepsilon\cos\theta}$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} \qquad \frac{dr(\theta)}{d\theta} = +\frac{k\,\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}$$

$$= \frac{\varepsilon}{k}\sin\theta\, r^2\frac{d\theta}{dt} = \frac{\varepsilon}{k}h\sin\theta \qquad = \frac{\varepsilon}{k}\sin\theta\, r^2$$

$$h(t) = r^2\frac{d\theta}{dt}$$

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \vec{v}\right|$$

$$\vec{r} = r\,\vec{e}_r$$

$$\vec{v} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{const.}$$

$$\vec{r} \times \vec{v} = r\frac{dr}{dt}\underbrace{\vec{e}_r \times \vec{e}_r}_{} = 0$$

$$+ r^2 \frac{d\theta}{dt}\underbrace{\vec{e}_r \times \vec{e}_\theta}_{|\vec{e}_r \times \vec{e}_\theta| = 1}$$

$$\Rightarrow \quad h = \text{const.}$$

$$\frac{dr}{dt} = \frac{\varepsilon}{k} h \sin\theta \qquad \frac{d^2r}{dt^2} = \frac{\varepsilon}{k} h \cos\theta \frac{d\theta}{dt}$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{\varepsilon}{k} h \cos\theta \frac{d\theta}{dt} - r\left(\frac{d\theta}{dt}\right)^2$$

$$h = r^2\frac{d\theta}{dt} \qquad \frac{d\theta}{dt} = \frac{h}{r^2}$$

$$\underbrace{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2}_{} = \frac{\varepsilon h^2}{k r^2}\cos\theta - r\frac{h^2}{r^4}$$

$$\frac{\vec{F}}{m} = \quad \downarrow \quad \vec{e}_r = \frac{h^2}{r^2}\left(\frac{\varepsilon}{k}\cos\theta - \frac{1}{r}\right)\vec{e}_r$$

$$r(\theta) = \frac{k}{1 + \varepsilon\cos\theta}$$

$$\frac{1}{r} = \frac{1}{k} + \frac{\varepsilon}{k}\cos\theta \Rightarrow \frac{\varepsilon}{k}\cos\theta - \frac{1}{r} = -\frac{1}{k}$$

$$\boxed{\vec{F} = -\frac{h^2}{k r^2} m\,\vec{e}_r}$$

3. Keplersche Gesetz $T^2/a^3 = \text{const.}$

(für alle Planeten des Sonnensystems)

$$\frac{dA}{dt} = \frac{1}{2} h$$

T : periode (Umlaufzeit)

(Flächengeschwindigkeit)

$$k = \frac{b^2}{a}$$

$$\text{Fläche} = \frac{dA}{dt} T = \pi a b = \pi \sqrt{a^3 k}$$
$$= \pi a \sqrt{a k}$$

$$T \cdot \frac{1}{2} h = \pi \sqrt{a^3 k}$$

$$T^2 h^2 = 4\pi^2 a^3 k$$

$$\Rightarrow \frac{T^2}{a^3} = 4\pi^2 \frac{k}{h^2} \qquad \vec{F} = -\frac{h^2}{k} \frac{m}{r^2} \vec{e}_r$$

$\Rightarrow \frac{h^2}{k}$ ist konstant für alle Planeten des Sonnensystems!

M

$$\frac{h^2}{k} = G M$$

$$\boxed{\vec{F} = -G \frac{M m}{r^2} \vec{e}_r}$$

$$\vec{F}_{Apfel} = -G \frac{M_{erde}\, m_{Apfel}}{R^2_{erde}} \vec{e}_r = -m_{Apfel}\, g\, \vec{e}_r$$

$$g \cong 9.81 \frac{m}{s^2} \quad \Longrightarrow \quad G \simeq (7 \pm 1)\, 10^{-11} N \left(\frac{m}{kg}\right)^2$$

Cavendish: 1798 (1% entfernt vom heute bekannten Wert)

$$G = 6.674 \times 10^{-11} N (m/kg)^2$$