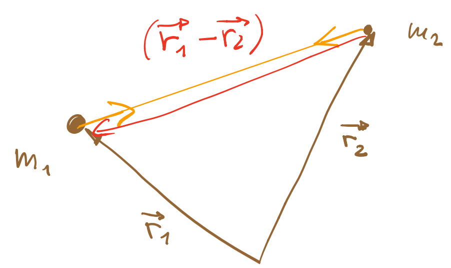
Lösung d. Bewegungsgleichung für ein Zweikörperproblem

Gravitationskraft des Zweikörperproblems

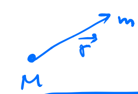


$$F_\alpha(\vec{r}) = -\frac{G\,m\,M}{r^2}\,\frac{x_\alpha}{r}$$

$$\vec{F}(\vec{r}) = -\frac{G\,m\,M}{r^2}\,\frac{\vec{r}}{r}$$

bisher

$$\vec{F} = -G\,\frac{m\,M}{r^2}\,\frac{\vec{r}}{r}$$



$$\boxed{V(\vec{r}) = -\frac{G\,m\,M}{r}}$$

$$F_\alpha = -\frac{\partial V(r)}{\partial x_\alpha}$$

$$= -\frac{\partial V(r)}{\partial r}\,\frac{\partial r}{\partial x_\alpha}$$

$$= -\frac{G\,m\,M}{r^2}\,\frac{\partial r}{\partial x_\alpha}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\,\frac{2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

Gravitationspotential des Zweikörperproblems

$$\boxed{V(|\vec{r}_1 - \vec{r}_2|) = -G\,\frac{m_1\,m_2}{|\vec{r}_1 - \vec{r}_2|}}$$

Kraft, die auf Himmelskörper 1 wirkt

$$\vec{F}_1 = -\frac{\partial V(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_1}$$

Kraft ——— " ——— 2 — " ———

$$\vec{F}_2 = -\frac{\partial V(\vec{r}_1 - \vec{r}_2)}{\partial \vec{r}_2} = -\vec{F}_1$$

Bewegungsgleichungen

$$m_1\,\frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1 \qquad m_2\,\frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2$$

übergang zu relativ und Schwerpunkt koordinaten

$$\vec{r} \equiv \vec{r}_1 - \vec{r}_2 \qquad \vec{R} \equiv \frac{m_1}{m_1+m_2}\vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}_2$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2\vec{r}_1}{dt^2} - \frac{d^2\vec{r}_2}{dt^2} = \frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2} = -\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\frac{\partial V(\vec{r})}{\partial \vec{r}}$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

reduzierte Masse

$$\boxed{\mu \frac{d^2\vec{r}}{dt^2} = -\frac{\partial V(\vec{r})}{\partial \vec{r}}} \qquad \mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$$

$$V(\vec{r}) = -\frac{G m_1 m_2}{r} = -G\frac{\mu M}{r}$$

totale Masse $M = m_1 + m_2$

hätten wir ein Potential

$$W(\vec{r}_1, \vec{r}_2) = f(\vec{r}, \vec{R}) \Rightarrow$$

die Dynamik d. relativen Koord. wird von $\vec{R}$ mit bestimmt

$$\frac{d^2\vec{R}}{dt^2} = \frac{m_1}{m_1+m_2}\frac{d^2\vec{r}_1}{dt^2} + \frac{m_2}{m_1+m_2}\frac{d^2\vec{r}_2}{dt^2}$$

$$= \frac{1}{m_1+m_2}\left(\vec{F}_1 + \vec{F}_2\right) = 0 \qquad \left(\text{da } \vec{F}_1 = -\vec{F}_2\right)$$

$\vec{R}$ - Dynamik ist triviale

$$\boxed{\mu \frac{d^2\vec{r}}{dt^2} = -\frac{\partial V}{\partial \vec{r}}} \qquad V(\vec{r}) = -G\frac{\mu M}{r}$$

Lösung der Relativbewegung $\qquad V(r) = -G\,\frac{\mu M}{r}$

Isotropes Problem
(es gibt keine Vorzugsrichtung)

$\Rightarrow \vec{F} \sim \vec{r} \Rightarrow \vec{L} = \vec{r} \times \vec{p} \quad : \quad \frac{d\vec{L}}{dt} = 0$

$\Rightarrow$ Bewegung in einer Ebene

O.E.d.A. legen wir diese Ebene in die x-y Ebene des Koordinatensystems

$$\vec{r} = r\,(\cos\theta,\ \sin\theta,\ 0)$$

$$\vec{r} = r\,\vec{e}_r \qquad \vec{e}_r = \cos\theta\,\vec{e}_x + \sin\theta\,\vec{e}_y$$

$$\vec{v} = \frac{dr}{dt}\,\vec{e}_r + r\,\frac{d\theta}{dt}\,\vec{e}_\theta \qquad \vec{e}_\theta = -\sin\theta\,\vec{e}_x + \cos\theta\,\vec{e}_y$$

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{e}_r + \left(2\,\frac{dr}{dt}\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right)\vec{e}_\theta$$

Drehimpuls $\qquad \vec{L} = \mu\,\vec{r} \times \vec{v} = \mu\,r^2\,\frac{d\theta}{dt}\,\vec{e}_z$

$L = |\vec{L}| = \mu\,r^2\,\frac{d\theta}{dt}$ ist konstant

$$\Rightarrow \frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\boxed{\mu\left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right) = -G\,\frac{\mu M}{r^2}}$$

Kraftkomponenten $\parallel \vec{e}_r$

$r(t) \qquad r(\theta(t))$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{\mu r^2}\frac{dr}{d\theta}$$

neue Koordinate $\quad s = \frac{1}{r}$

$$\frac{dr}{d\theta} = -\frac{1}{s^2}\frac{ds}{d\theta} \Rightarrow \frac{dr}{dt} = -\frac{L}{\mu}\frac{ds}{d\theta}$$

$$= -\frac{L}{\mu}\frac{s^2}{s^2}\frac{ds}{dt}$$

$$\frac{d^2r}{dt^2} = -\frac{L}{\mu}\frac{d^2s}{d\theta\, dt}$$

$$= -\frac{L}{\mu}\frac{d^2s}{d\theta^2}\frac{d\theta}{dt} = -\left(\frac{L}{\mu}\right)^2 s^2 \frac{d^2s}{d\theta^2}$$

$$\mu\left(-\left(\frac{L}{\mu}\right)^2 s^2 \frac{d^2s}{d\theta^2} - s^3\left(\frac{L}{\mu}\right)^2\right) = -G\mu M s^2$$

$$\Rightarrow \quad \underline{\frac{d^2s}{d\theta^2} + s = s_0} \qquad s_0 = \frac{G}{L^2}\mu^2 M$$

wie die DGl des harmonischen Oszillators

$$s_{hom} = C\cos(\theta + \theta_0) \qquad \text{wählen } \theta_0 = 0$$

$$s_{inhom} = s_0$$

$$s(\theta) = C\cos(\theta) + s_0$$

$$\Rightarrow \quad r(\theta) = \frac{k}{1 + \varepsilon\cos\theta} \qquad k = \frac{1}{s_0} = \frac{L^2}{G\mu^2 M}$$

$$\varepsilon = \frac{C}{s_0} = \frac{C L^2}{G M \mu^2}$$

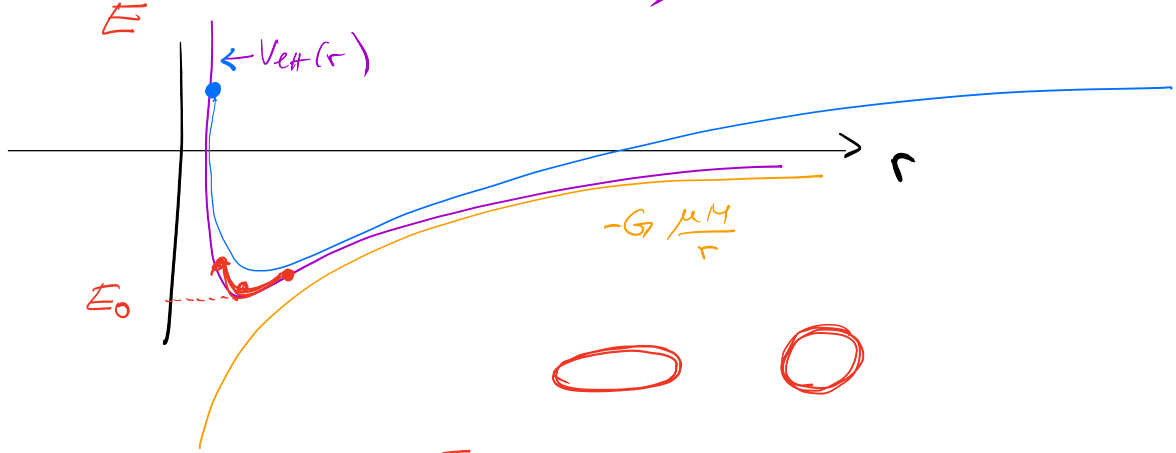
Parametrisierung einer Ellipse

Bestimmung d. Integrationskonstanten

$$E = \frac{\mu}{2}\left(\frac{d\vec{r}}{dt}\right)^2 + V(r) \qquad \frac{d\vec{r}}{dt} = \frac{dr}{dt}\,\vec{e}_r + r\frac{d\theta}{dt}\,\vec{e}_\theta$$

$$= \frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 + \frac{\mu}{2} r^2\left(\frac{d\theta}{dt}\right)^2 + V(r)$$

$$= \frac{\mu}{2}\left(\frac{dr}{dt}\right)^2 + \underbrace{\frac{L^2}{2\mu r^2} - G\frac{\mu M}{r}}_{V_{eff}(r)}$$



$E_0 < E < 0 \Rightarrow$ Ellipsen

$E > 0$ -

$$E_0 = -\frac{G^2\mu^3 M^2}{2L^2}$$

$$\varepsilon = \sqrt{\frac{E - E_0}{|E_0|}} \Rightarrow E_0 < E < 0$$

$\Rightarrow 0 < \varepsilon < 1$

$\Rightarrow$ Ellipse

$E > 0 \Rightarrow \varepsilon > 1$