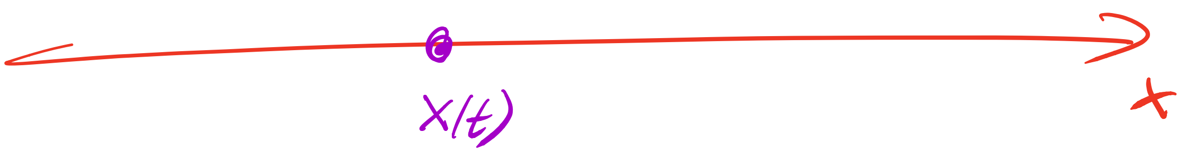
Bewegungsgleichung d. Mechanik

$$\boxed{m \frac{d^2 \vec{r}}{dt^2} = \vec{F}(\vec{r}, t)}$$

Energieerhaltung in einer Dimension



$$m \frac{d^2 x}{dt^2} = F(x, t)$$

$g(t) \quad : \quad \frac{d g(t)}{dt}$

$g^2(t) \qquad \frac{d}{dt} g^2(t) = 2 g(t) \frac{dg}{dt}$

äußere Ableitung

innere Ableitung

$$\frac{d g^2}{dg} = 2g$$

$$\frac{d}{dt}\, g^2(t) = \frac{d g^2(t)}{dg}\, \frac{dg}{dt}$$

$$\lim_{\Delta t \to 0} \frac{g^2(t+\Delta t) - g^2(t)}{\Delta t}$$

$$m \frac{d^2 x}{dt^2} = F(x,t)$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = \frac{dx}{dt}\,\frac{d^2x}{dt^2}$$

$$m \frac{dx}{dt}\,\frac{d^2x}{dt^2} = F(x,t)\,\frac{dx}{dt}$$

$$\boxed{\frac{m}{2}\frac{d}{dt}\left(\frac{dx}{dt}\right)^2 = F(x,t)\,\frac{dx}{dt}}$$

$$\int_{t_0}^{t} \frac{df(t')}{dt'}\, dt' = f(t) - f(t_0)$$

$$\int_{t_0}^{t} \frac{m}{2} \frac{d}{dt'} \Big(\underbrace{\frac{dx}{dt'}}_{v(t')} \Big)^2 dt' = \int_{t_0}^{t} F(x,t') \frac{dx}{dt'}\, dt'$$

$$\frac{m}{2} \left(v^2(t) - v^2(t_0) \right) = \int_{t_0}^{t} F(x,t') \frac{dx}{dt'}\, dt'$$

$$\boxed{\text{Annahme} \quad F(x,t) = F(x)}$$

$$I = \int_{u_0}^{u_1} f(u)\, du$$

$$u(s)$$

$$u(s_0) = u_0$$

$$u(s_1) = u_1$$

$$= \int_{s_0}^{s_1} f(u(s)) \frac{du}{ds}\, ds$$

$$\frac{m}{2}\left(v^2(t) - v^2(t_0)\right) = \int_{x_0}^{x} F(x')\,dx' = -\int_{x_0}^{x} \frac{dU(x')}{dx'}\,dx'$$

$$= -\left(U(x) - U(x_0)\right)$$

$$\boxed{F(x) = -\frac{dU(x)}{dx}}$$

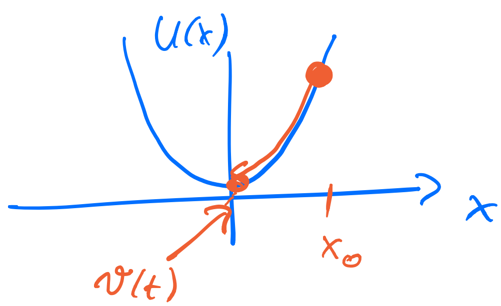
$x = x(t)$

$x_0 = x(t_0)$

$$E = \underbrace{\frac{m}{2} v^2(t)}_{\text{kinetische Energie}} + \underbrace{U(x(t))}_{\text{potentielle Energie}} = \frac{m}{2} v^2(t_0) + U(x(t_0))$$

Energie ist erhalten !

$$F(x) = -kx \qquad U(x) = \frac{k}{2}x^2 + \cancel{U_0}$$

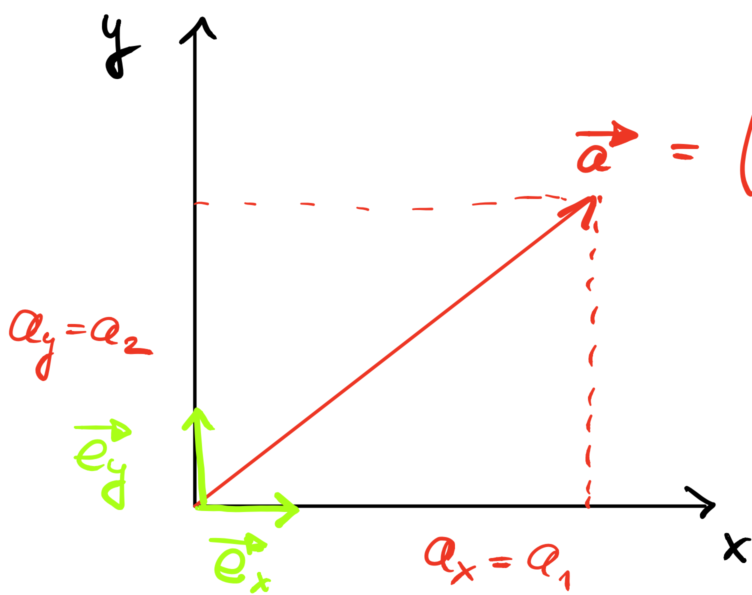


$$U(x_0) = \frac{k}{2}x_0^2$$

$$\frac{m}{2}v(t)^2 = \frac{k}{2}x_0^2$$

Energieerhaltung ist eine Konsequenz der Homogenität d. Zeit

Vektoren, Ableitungen, Polarkoordinaten



$$\vec{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{a} = a_x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + a_y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_x \vec{e}_x + a_y \vec{e}_y$$

$$= \sum_{i=1}^{2} a_i \vec{e}_i$$

$$\vec{a} = \sum_{i=1}^{3} a_i \vec{e}_i$$

$$\vec{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

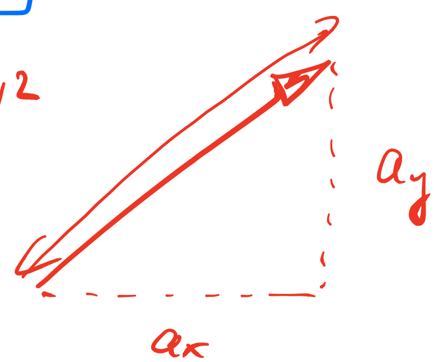
$$\vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

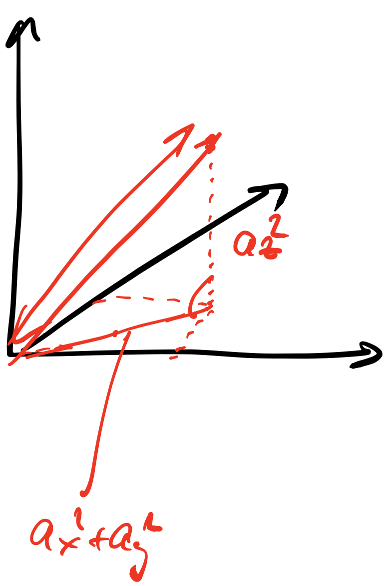
Skalarprodukt:

$$\vec{a} \cdot \vec{b} \equiv \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$$

koordinatenunabhängig

$$\vec{a} \cdot \vec{a} = \sum_{i=1}^{3} a_i^2 = |\vec{a}|^2$$

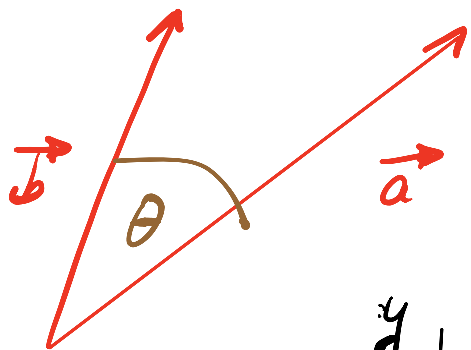




$$a_x^2 + a_y^2 = (\text{Länge des Vektors})^2$$

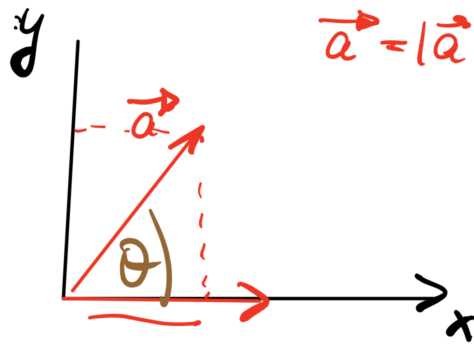
$|\vec{a}|$: Länge des Vektors

$$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|\ \cos(\theta)$$



θ : theta

$$\vec{a} = |\vec{a}| \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



$$\vec{b} = \begin{pmatrix} b_x \\ 0 \end{pmatrix}$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y = |\vec{a}| \cos\theta\ |\vec{b}|$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos(\theta)$$

$$\vec{e}_i \cdot \vec{e}_j = 1 \quad \text{wenn} \quad i = j$$
$$= 0 \quad i \neq j$$

$$= \delta_{i,j} \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Vektorprodukt:

$$\vec{c} = \vec{a} \times \vec{b}$$

$$c_i = \sum_{jk=1}^{3} \varepsilon_{ijk}\, a_j\, b_k$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \{i,j,k\} = \{1,2,3\} \\ -1 & \{i,j,k\} = \{2,1,3\} \\ 0 & \text{andernfalls} \end{cases}$$

od. cycl. permut.

$\{1,2,3\} \quad \{3,1,2\}, \{2,3,1\}$

$$c_x = a_y b_z - a_z b_y$$

$$c_y = a_z b_x - a_x b_z$$

$$c_z = a_x b_y - a_y b_x$$

$$\vec{c} \cdot \vec{a} = \sum_i c_i a_i$$

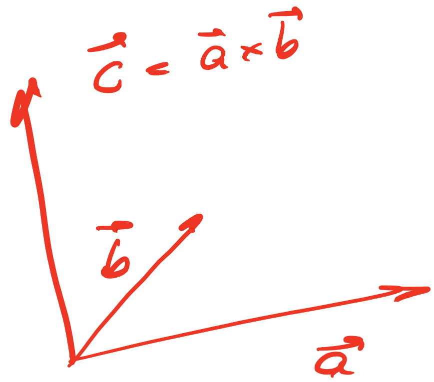
$$= \sum_{ijk} \underline{\varepsilon_{ijk}} a_j b_k a_i$$

$$= \sum_{jik} \underline{\varepsilon_{jik}} a_i b_k a_j$$

$$\varepsilon_{ijk} = -\varepsilon_{jik}$$

$$= -\sum_{ijk} \varepsilon_{ijk} a_j b_k a_i = -\vec{c} \cdot \vec{a}$$

$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{a} = 0}$$



$|\vec{c}|$?

$$\sum_{i=1}^{3} \varepsilon_{ijk}\, \varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}$$

(Übungsaufgabe)

$$|\vec{c}|^2 = \sum_i c_i^2 = \sum_{ijklm} \varepsilon_{ijk}\, \varepsilon_{ilm}\, a_j b_k a_l b_m$$

$$= \sum_{jklm} \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{lk}\right) a_j a_l b_k b_m$$

$$= \sum_{jk} a_j a_j b_k b_k - \sum_{jk} a_j b_j a_k b_k$$

$$= |\vec{a}|^2 |\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2$$

$$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$|\vec{c}| = |\vec{a}|\,|\vec{b}|\,|\sin\theta|$$

$$\vec{a}\times\vec{b} = \sum_{ijk} \varepsilon_{ijk}\, \underline{a_j\, b_k}\, \vec{e_i}$$

$$\uparrow$$

$$= -\sum_{ijk} \varepsilon_{ijk}\, b_j\, a_k\, \vec{e_i}$$

$$= -\,\vec{b}\times\vec{a}$$
