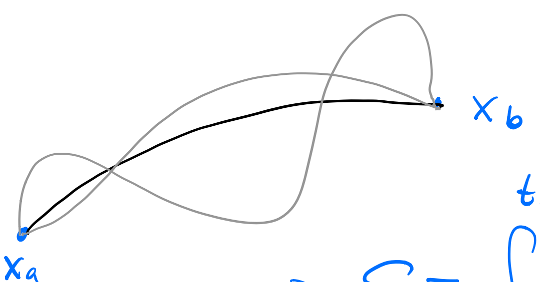
Lagrange Funktion $L(x,\dot{x},t)$

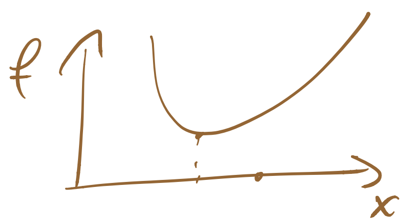


Wirkung $\rightarrow$ $S = \int_{t_a}^{t_b} L(x(t),\dot{x}(t),t)\,dt$

Der physikalisch realisierte Weg sei $x(t)$.

$x(t)$ ergibt ein Extremum von S

Funktion $f(x)$

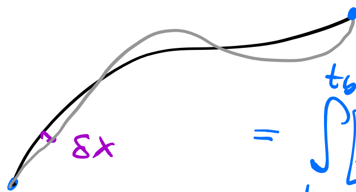


$x \in \mathbb{R} \rightarrow f \in \mathbb{R}$

Funktional $\quad x(t) \longrightarrow S \quad S[x(t)]$

Wenn $x(t)$ S extremiert, dann betrachten wir $\underbrace{x(t)+\delta x(t)}$: $\left.\begin{array}{l} x(t_a)=x_a \\ x(t_b)=x_b \end{array}\right\} \Rightarrow \begin{array}{l} \delta x(t_a)=0 \\ \delta x(t_b)=0 \end{array}$

$\frac{d}{dt}(\ ) = \dot{x}(t) + \delta\dot{x}(t)$



$$\Delta S = S[x+\delta x] - S[x]$$

$$= \int_{t_a}^{t_b} \left[L(x+\delta x, \dot{x}+\delta\dot{x}, t) - L(x,\dot{x},t) \right] dt$$

$$L(x+\delta x, \dot{x}+\delta\dot{x}, t) \simeq L(x,\dot{x},t) + \underline{\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta\dot{x}} + \dots$$

$$\Delta S = \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial x}\delta x(t) + \underline{\frac{\partial L}{\partial \dot{x}}\delta\dot{x}(t)} \right) dt$$

$$\int_{t_a}^{t_b} \frac{\partial L}{\partial \dot{x}} \left(\frac{d}{dt}\delta x \right) dt$$

partielle Integration

•) $$\underline{\int_{t_a}^{t_b} \frac{d f(t)}{dt}\, dt = f(t_b) - f(t_a)}$$

:) $f(t) = u(t)\cdot v(t)$

$$\frac{df}{dt} = \frac{du}{dt}v + u\frac{dv}{dt}$$

$$\int_{t_a}^{t_b} \frac{df}{dt}\, dt = \int_{t_a}^{t_b} \frac{du}{dt}\cdot v\, dt + \int_{t_a}^{t_b} u\frac{dv}{dt}\, dt$$

$$= u(t_b)\, v(t_b) - u(t_a)\, v(t_a)$$

$$\int_{t_a}^{t_b} \frac{du}{dt}\, v \cdot dt = -\int_{t_a}^{t_b} u \,\frac{dv}{dt}\, dt + u(t_b)\,v(t_b) - u(t_a)\,v(t_a)$$

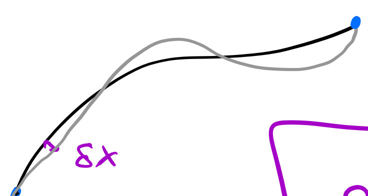
$$\int_{t_a}^{t_b} \frac{\partial L}{\partial \dot x}\left(\frac{d}{dt}\,\delta x\right) dt = -\int_{t_a}^{t_b}\left(\frac{d}{dt}\,\frac{\partial L}{\partial \dot x}\right)\delta x \; dt \; \underbrace{+\left(\frac{\partial L}{\partial \dot x}\,\delta x\right)(t_b) - \left(\frac{\partial L}{\partial \dot x}\,\delta x\right)(t_a)}_{0}$$

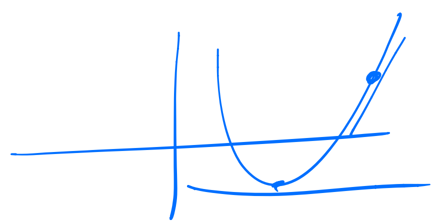
$v = \frac{\partial L}{\partial \dot x}$

$u = \delta x$

$$\Delta S = \int_{t_a}^{t_b}\left(\frac{\partial L}{\partial x}\,\delta x(t) + \frac{\partial L}{\partial \dot x}\,\delta \dot x(t)\right) dt$$

$$= \int_{t_a}^{t_b} \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot x}\right)}_{= 0} \underline{\underline{\delta x}} \; dt$$





$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\,\frac{\partial L}{\partial \dot x} = 0}$$

Euler-Lagrange Gleichung

Ist die Funktion $L(x, \dot{x}, t)$ eindeutig bestimmt?

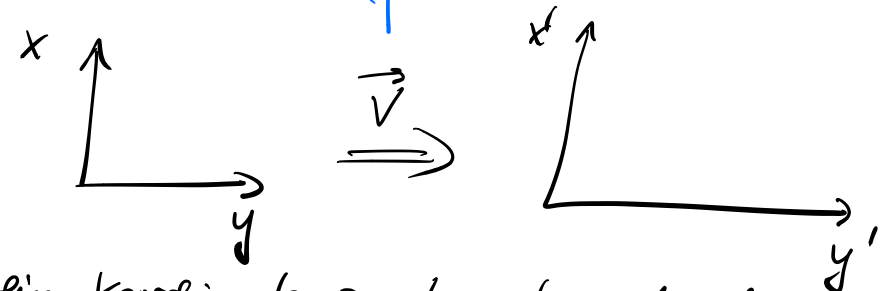
$$L \to L' = L + \frac{d}{dt} f(x,t)$$

$$S' = \int_{t_a}^{t_b} L'(x,\dot{x},t)\,dt = S + \int_{t_a}^{t_b} \frac{d}{dt} f(x,t)\,dt$$

$$= S + f(x_b,t_b) - f(x_a,t_a)$$

S, S' sind gleichwertig

Galilei Transformation



ein Koordinatensystem bewegt sich relativ zum anderen mit konstanter Geschwindigkeit

$$\vec{r} = \vec{r}\,' + \vec{V} t \qquad t' = t$$

$$\dot{\vec{r}} = \dot{\vec{r}}\,' + \vec{V} \qquad \text{(Zeit ist absolut)}$$

$\boxed{\text{Galilei Transformation}}$

•) räumlich + zeitlich homogenes Problem

$L(x, \dot{x}, t)$ 3d: $L(\vec{x}, \dot{\vec{x}}, t)$

$= L(\vec{x}, \vec{v}, t)$

$v = |\vec{v}|$ $= L(\vec{v}^{\,2}) = L(v^2)$

$$L' = L(v'^2) = L\left((\vec{v} - \vec{V})^2\right)$$

$$= L\left(\vec{v}^2 - 2\vec{v}\cdot\vec{V} + \cancel{\vec{V}^2}\right)$$

$$= L(v^2) - 2\vec{v}\cdot\vec{V}\,\frac{\partial L}{\partial v^2} + \ldots$$

$$\vec{v}\,\frac{\partial L}{\partial v^2} = \frac{d}{dt}\vec{f}(\vec{x},t)$$

$$\frac{d}{dt}\vec{f}(\vec{x},t) = \sum_{i=1}^{d}\frac{\partial\vec{f}}{\partial x_i}v_i + \frac{\partial\vec{f}}{\partial t}$$

$\frac{\partial L}{\partial v^2}$ ist unabhängig von v

$$\frac{\partial L}{\partial v^2} = \text{const.}$$

$$\Rightarrow\quad L = \frac{m}{2}v^2 = \frac{m}{2}\vec{v}\cdot\vec{v}$$

$$= \frac{m}{2}\sum_{i=1}^{3}\dot{x}_i\,\dot{x}_i$$

$$\underline{\frac{\partial L}{\partial x_i}} - \underline{\underline{\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}}} = 0$$

$$\frac{\partial L}{\partial \dot{x}_j} = m\,\dot{x}_j$$

$$\boxed{m\,\frac{d\dot{x}_i}{dt} = 0}$$

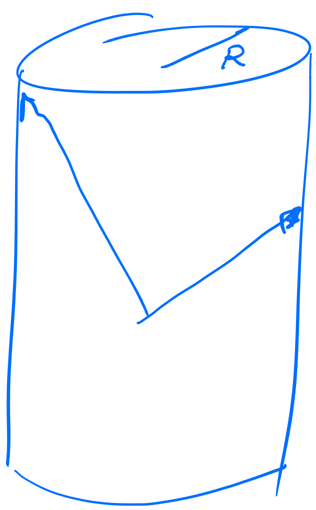
1. Newton'sche Gesetz

Lagrange Funktion mit Kraft

$$L(\vec r, \dot{\vec r}, t) = \underbrace{\frac{m}{2}\vec v^2}_{\text{kinetische Energie}} - U(\vec r)$$

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot x_i} = -\frac{\partial U}{\partial x_i} - m\ddot x_i = 0$$

$$m\ddot{\vec x} = \vec F = -\nabla U(\vec r)$$

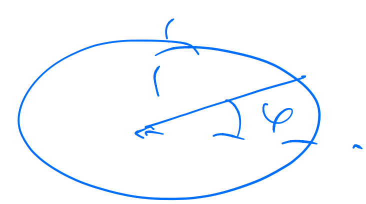


Bewegung auf der Oberfläche eines Zylinders

$$\vec r(t) = x(t)\,\vec e_x + y(t)\,\vec e_y + z(t)\,\vec e_z$$

$$x^2(t) + y^2(t) = R^2$$

$$x(t) = R\cos\varphi(t)$$
$$y(t) = R\sin\varphi(t)$$



$$\frac{d\vec r}{dt} = \dot x\,\vec e_x + \dot y\,\vec e_y + \dot z\,\vec e_z$$
$$= \underline{\left(-\sin\varphi(t)\,\vec e_x + \cos\varphi(t)\,\vec e_y\right)}\,R\dot\varphi + \dot z\,e_z$$

$$T = \frac{m}{2}\vec v^2 = \underline{\frac{m}{2}\dot z^2 + \frac{m}{2}R^2\dot\varphi^2}$$

$$\underbrace{\left(\sin^2\varphi + \cos^2\varphi\right)}_{1} R^2 \dot{\varphi}^2$$

$$L = \frac{m}{2}\dot{z}^2 + \frac{m}{2}R^2\dot{\varphi}^2 - U(\varphi, z)$$

$$\frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0 \qquad\qquad \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0$$

$$\boxed{m\ddot{z} = -\frac{\partial U}{\partial z}} \qquad\qquad \boxed{mR^2\ddot{\varphi} = -\frac{\partial U}{\partial \varphi}}$$