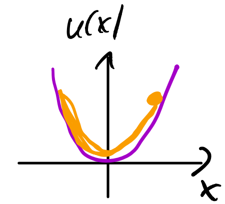
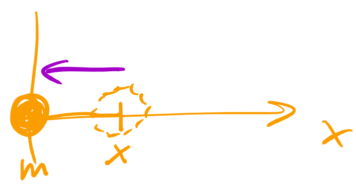
Mechanische Bewegung in einer Dimension

1. Der harmonische Oszillator



Potential : $U(x) = \frac{k}{2} x^2$

Kraft: $F(x) = -\frac{\partial U}{\partial x} = -kx$



$U(x,t) = U(x)$: Energieerhaltung

Bewegungsgleichung:

$$-kx = F = ma = m \frac{d^2x}{dt^2}$$

$$\boxed{m \frac{d^2x}{dt^2} + kx = 0}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$\frac{k}{m}$: (Inverse Zeit)^2

es gibt eine Zeit

$T = \sqrt{\frac{m}{k}}$

$$\frac{d^2x}{dt^2} = -\frac{k}{m} x$$

$$\frac{d}{dz}\cos(z) = -\sin(z) \qquad \frac{d}{dz}\sin(z) = \cos(z)$$

$$\frac{d^2}{dz^2}\cos(z) = -\cos(z) \qquad \frac{d^2}{dz^2}\sin(z) = -\sin(z)$$

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

$$\frac{dx(t)}{dt} = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$

$$\frac{d^2x(t)}{dt^2} = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)$$

$$= -\omega^2\underbrace{\left(a\cos(\omega t) + b\sin(\omega t)\right)}_{x(t)}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad \Longrightarrow \qquad \boxed{\omega^2 = \frac{k}{m}}$$

$$\cos(\omega[t+T]) = \cos(\omega t)$$

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$x(t) = x_0\cos(\omega t) + \frac{v_0}{\omega}\sin(\omega t)$$

$$x_0 = x(t=0) \qquad v_0 = v(t=0)$$

$$x(t) = a\cos(\omega t) + b\sin(\omega t)$$

$$x(t=0) = a\,1 + b\cdot 0 = a = x_0$$

$$v(t) = \frac{dx(t)}{dt} = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$

$$v(t=0) = -a\omega\cdot 0 + b\omega\,1 = b\omega = v_0$$

komplexe Zahlen

Frage: was ist die Wurzel von -4?

$$\sqrt{-4} = \sqrt{-1}\,\sqrt{4} = \sqrt{-1}\,2$$

$$\sqrt{-1} = i$$

$$i = \sqrt{-1} \qquad i^2 = \sqrt{-1}\,\sqrt{-1} = -1$$

$$\frac{1}{i} = \frac{i}{i}\cdot\frac{1}{i} = -i$$

$$z = x + iy \qquad x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z)$$

x: Realteil, y: Imaginärteil

$$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$$

$$\Rightarrow z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$$

$z = x + iy$ $\qquad$ $z^* = x - iy$

komplex konjugierte

$z^* z = x^2 + y^2$

Funktionen mit komplexen Argumenten:

? $\quad e^z = e^{x+iy}$ $\qquad$ $(e^{a+b} = e^a \cdot e^b)$

$= e^x \cdot e^{iy}$

$\frac{d}{dy} e^{\lambda y} = \lambda e^{\lambda y} \Rightarrow \frac{d}{dy} e^{iy} = i e^{iy}$

$\frac{d^2}{dy^2} e^{iy} = - e^{iy}$

$e^{iy} = a \cos y + b \sin y$

$\frac{d}{dy} e^{iy} = i (a \cos y + b \sin y)$

$= - a \sin y + b \cos y$

$\Rightarrow \quad ia = b$

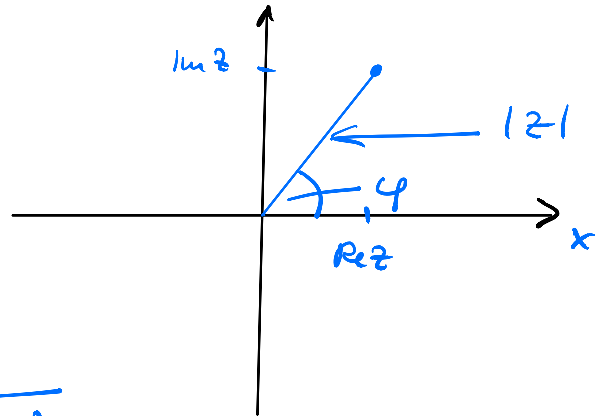
$(\times i) \quad ib = -a$

$-b = -ia$

$y = 0 \qquad e^{iy}\Big|_{y=0} = e^0 = 1 = a$

$e^{iy} = \cos y + i \sin y$

$z = x + iy$



$z^* z = x^2 + y^2$

$|z| \equiv \sqrt{z^* z} = \sqrt{x^2+y^2}$

$$z = x + iy = |z| (\cos\varphi + i \sin\varphi) = |z| e^{i\varphi}$$

$$\frac{x}{|z|} = \cos\varphi \qquad \frac{y}{|z|} = \sin\varphi \quad \Rightarrow \quad \frac{y}{x} = \tan\varphi$$

$$\varphi = \arctan\left(\frac{y}{x}\right)$$

Differentialgleichungen

$$y^{(n)} \equiv \frac{d^n y}{dx^n}$$

$$y^{(n)}(x) + a_{n-1}\, y^{(n-1)}(x) + \dots + a_0\, y(x) = 0$$

-) linear d.h. $A y_1(x) + B y_2(x)$ ist Lösung, wenn $y_1(x)$ und $y_2(x)$ Lösungen sind
-) homogen $\dots = 0$
-) konstante Koeffizienten $a_{n-1}, a_{n-2} \dots a_0$
-) n-ter Ordnung (größte Ableitung n)

$$y(x) = y_0 \; e^{\lambda x}$$

$$y^{(1)}(x) = y_0 \, \lambda \, e^{\lambda x} \qquad y^{(2)}(x) = y_0 \, \lambda^2 e^{\lambda x}$$

$$y^{(n)}(x) = y_0 \, \lambda^n e^{\lambda x}$$

$$y^{(n)}(x) + a_{n-1} \, y^{(n-1)}(x) + \dots + a_0 \, y(x) = 0$$

$$y_0 \lambda^n e^{\lambda x} + a_{n-1} y_0 \, \lambda^{n-1} e^{\lambda x} + \dots + a_0 \, y_0 e^{\lambda x} = 0$$

$$\lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_0 = 0$$

Nullstellen des charakteristischen Polynoms

Es gibt genau n solcher Nullstellen λ_i

$$y(x) = \sum_{j=1}^{n} y_{0_j} \; e^{\lambda_j x}$$

zu bestimmen durch die Anfangsbedingungen

Wenn zwei Nullstellen identisch sind

$$\lambda_k = \lambda_e$$

$$y_{0k} \, e^{\lambda_k x} + x \, y_{0e} \, e^{\lambda_e x}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

$$x = x_0 e^{\lambda x}$$

$$\lambda^2 + \frac{k}{m} = 0 \qquad \lambda = \pm i \sqrt{\frac{k}{m}}$$

$$= \pm i \omega$$

$$x_\pm(t) = x_{\pm,0} \quad e^{\pm i\omega t}$$

$$x(t) = x_{+,0}\, e^{i\omega t} + x_{-,0}\, e^{-i\omega t}$$

$$x(t)^* = x_{+,0}^*\, e^{-i\omega t} + x_{-,0}^*\, e^{i\omega t}$$

$$x(t) = x^*(t) \Rightarrow x_{\pm,0} = x_{\mp,0}^*$$