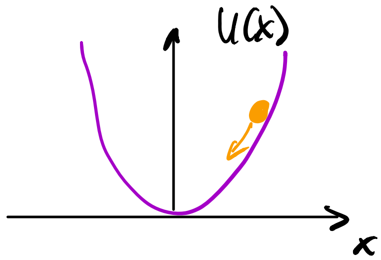
harmonische Oszillator



$$U(x) = \frac{k}{2} x^2 \qquad F(x) = -\frac{\partial U(x)}{\partial x} = -kx$$

$$m \frac{d^2 x(t)}{dt^2} = F(x) = -k\,x(t)$$

$$\boxed{\frac{d^2 x(t)}{dt^2} + \frac{k}{m} x(t) = 0}$$

$$x(t) = x_0 e^{\lambda t}$$

$$\frac{dx(t)}{dt} = \lambda x(t)$$

$$\frac{d^2 x}{dt^2} = \lambda^2 x$$

$$\lambda^2 + \frac{k}{m} = 0$$

$$\lambda = \pm\sqrt{-\frac{k}{m}} = \pm i \sqrt{\frac{k}{m}} \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\boxed{x(t) = x_+ e^{i\omega_0 t} + x_- e^{-i\omega_0 t}}$$

$$z = x + iy$$
$$z^* = x - iy$$

$$x(t) = x^*(t) = x_+^* e^{-i\omega_0 t} + x_-^* e^{i\omega_0 t}$$

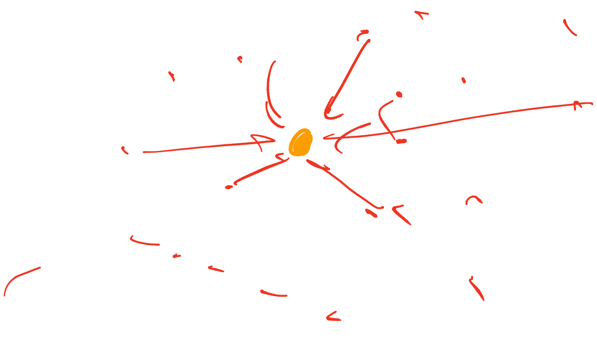
$$x_+^* = x_- \qquad x_+ = a + ib$$
$$x_- = a - ib$$

$$x(t) = a\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) + ib\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \qquad e^{-i\varphi} = \cos\varphi - i\sin\varphi$$

$$\boxed{x(t) = 2a\cos(\omega_0 t) - 2b\sin(\omega_0 t)}$$

Oszillation mit Reibung



Reibungskraft

$$F = -\Gamma \frac{dx}{dt}$$

$$m\frac{d^2x}{dt^2} + \Gamma\frac{dx}{dt} + k\,x = 0$$

$$x(t) = x_0\, e^{\lambda t}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0 \qquad \gamma = \frac{\Gamma}{m}$$

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

(bisher $\lambda_{1,2}^{\gamma=0} = \pm i\omega_0$)

$$x(t) = x_1\, e^{-\frac{\gamma}{2}t}\, e^{+\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\,t} + x_2\, e^{-\frac{\gamma}{2}t}\, e^{-\sqrt{\frac{\gamma^2}{4}-\omega_0^2}\,t}$$

Oszillatorisches Verhalten wenn

$$\boxed{\frac{\gamma}{2} < \omega_0}$$

Gedämpftes Verhalten wenn

$$\boxed{\frac{\gamma}{2} > \omega_0}$$

$\gamma < 2\omega_0$

$$\lambda_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

$$\Omega_0 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

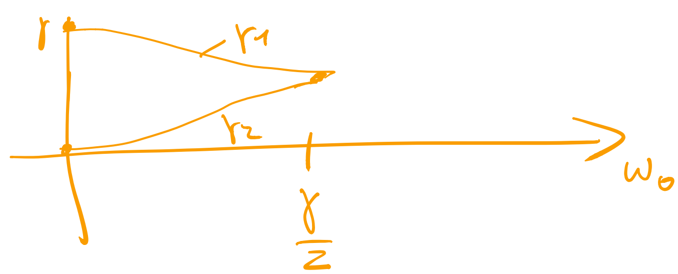
$$\lambda_{1,2} = -\frac{\gamma}{2} \pm i\,\Omega_0$$

$$x(t) = e^{-\gamma/2\, t}\left(a \cos \Omega_0 t + b \sin \Omega_0 t\right)$$

$\gamma > 2\omega_0$

$$\gamma_{1,2} = \frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}$$

$$x(t) = a\, e^{-\gamma_1 t} + b\, e^{-\gamma_2 t}$$



$\frac{\gamma}{2} = \omega_0$ dann sind die beiden unabhängigen Lösungen

$$x_1(t) = x_0\, e^{\lambda t}$$

$$x_2(t) = b\, t\, e^{\lambda t}$$

$$x(t) = a\, e^{-\gamma_1 t} + b\, e^{-\gamma_2 t}$$

$$x_0 = x(t=0) = a + b$$

$$v_0 = \left.\frac{dx}{dt}\right|_{t \to 0} = -\gamma_1 a - b \gamma_2$$

$$x(t) = \underbrace{\frac{1}{\gamma_2 - \gamma_1}}_{\frac{1}{2\varepsilon}} \underbrace{\left[(\gamma_2 x_0 + v_0)\, e^{-\gamma_1 t} - (\gamma_1 x_0 + v_0)\, e^{-\gamma_2 t} \right]}_{2\varepsilon\, e^{-\frac{\gamma}{2} t} \left(v_0 t + \frac{\gamma}{2} x_0 t + x_0 \right)}$$

$$\gamma_1 = \frac{\gamma}{2} - \varepsilon$$

$$\gamma_2 = \frac{\gamma}{2} + \varepsilon$$

$$\varepsilon = \sqrt{\frac{\gamma^2}{2^2} - \omega_0^2}$$

$$(\gamma_2 x_0 + v_0)\, e^{-\gamma_1 t} - (\gamma_1 x_0 + v_0)\, e^{-\gamma_2 t}$$

$$= \left(\left(\frac{\gamma}{2} + \varepsilon \right) x_0 + v_0 \right) e^{-\left(\frac{\gamma}{2} - \varepsilon \right) t} - \left(\left(\frac{\gamma}{2} - \varepsilon \right) x_0 + v_0 \right) e^{-\left(\frac{\gamma}{2} + \varepsilon \right) t}$$

$$=\left(\frac{\gamma}{2}x_0+v_0\right)e^{-\frac{\gamma}{2}}\cdot\left(e^{\varepsilon t}-e^{-\varepsilon t}\right)$$

$$+\ \varepsilon x_0 \qquad \left[e^{-\left(\frac{\gamma}{2}-\varepsilon\right)t}+e^{-\left(\frac{\gamma}{2}+\varepsilon\right)t}\right]$$

$$e^{x}=1+x+\frac{1}{2}x^2+\dots\ \frac{1}{n!}x^n$$

$$=\left(\frac{\gamma}{2}x_0+v_0\right)e^{-\frac{\gamma}{2}} \qquad 2\varepsilon t$$

$$+\ \varepsilon x_0 \qquad \left[e^{-\left(\frac{\gamma}{2}-\cancel{\varepsilon}\right)t}+e^{-\left(\frac{\gamma}{2}+\cancel{\varepsilon}\right)t}\right]$$

$$=\varepsilon\left[\left(\gamma x_0+2v_0\right)t+2x_0\right]e^{-\frac{\gamma}{2}t}$$

- getriebene Oszillatoren

$$m\frac{d^2x}{dt^2}=-kx-\Gamma\frac{dx}{dt}+F_{extern}(t)$$

$$\boxed{\frac{d^2x}{dt^2}+\omega_0^2x+\gamma\frac{dx}{dt}=f(t)} \qquad f=\frac{F_{ext}}{m}$$

inhomogene D.G.L.