$$m \frac{d^2x}{dt} = -kx - \Gamma \frac{dx}{dt} + F_{ext}(t)$$

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x + \gamma \frac{dx}{dt} = f(t)}$$

$\omega_0^2 = \frac{k}{m}$

$\gamma = \Gamma/m$

Inhomogenität $f = F_{ext}/m$

für $f = 0$ kennen wir die Lösung

$$x_{1,2}(t) = e^{-\frac{\gamma}{2}t} e^{\pm i \Omega t} \qquad \Omega = \sqrt{\omega_0^2 - \gamma^2/4} \qquad \frac{\gamma}{2} < \omega_0$$

$$x_{1,2}(t) = e^{-\gamma_{1,2} t} \qquad \gamma_{1,2} = \frac{\gamma}{2} \mp \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \qquad \frac{\gamma}{2} > \omega_0$$

die Lösungen der inhomogenen DGL kann man folgendermaßen konstruieren:

$$x(t) = \underbrace{C_1 x_1(t) + C_2 x_2(t)}_{\text{Lösungen der homogenen DGL.}} + \bar{x}(t)$$

irgendeine spezielle Lösung der inhomog. DGL

$$\hat{O}\, x(t) = \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x$$

$$\frac{d^2x}{dt^2} + \omega_0^2\, x + \gamma \frac{dx}{dt} = f(t) \qquad x(t) = C_1\, x_1(t) + C_2\, x_2(t) + \bar{x}(t)$$

$$\hat{O}\, x = f(t)$$

$$\hat{O}\, x = C_1 \underbrace{\hat{O}\, x_1}_{=0} + C_2 \underbrace{\hat{O}\, x_2}_{=0} + \underbrace{\hat{O}\, \bar{x}}_{=f} = f$$

Verbleibende Aufgabe: finde eine spezielle Lösung $\bar{x}$ der inhom. DGL

---

wenn $f(t) = f_1(t) + f_2(t)$

dann $\bar{x}(t) = \bar{x}_1(t) + \bar{x}_2(t)$

$$\hat{O}\,\bar{x}_1 = f_1 \qquad \hat{O}\,\bar{x}_2 = f_2$$

Welche Inhomogenität wäre denn einfach genug, so dass wir sie raten können?

$$f(t) = f_0 \cos(\omega t)$$

$$= \underbrace{\frac{f_0}{2} e^{i\omega t}}_{f_1} + \underbrace{\frac{f_0}{2} e^{-i\omega t}}_{f_2}$$

ich rate: $\bar{x}_1(t) = A f_0 e^{i\omega t}$

$$\frac{d^2\bar{x}_1}{dt^2} + \gamma \frac{d\bar{x}_1}{dt} + \omega_0^2 \bar{x}_1$$

$$= (-\omega^2 + i\gamma\omega + \omega_0^2) A f_0 \cancel{e^{i\omega t}} \stackrel{!}{=} \frac{f_0}{2} \cancel{e^{i\omega t}}$$

$$\boxed{A = \frac{1}{2} \frac{1}{-\omega^2 + i\gamma\omega + \omega_0^2}}$$

$$\bar{x}_2 = A(-\omega) f_0 e^{-i\omega t}$$

$$\bar{x}_1 = A(\omega) f_0 e^{i\omega t}$$

$$\Rightarrow \quad \bar{x}(t) = f_0 \left(A(\omega) e^{i\omega t} + A(-\omega) e^{-i\omega t} \right)$$

$$A(-\omega) = A(\omega)^*$$

$$\bar{x}(t) = 2 f_0 \operatorname{Re}\left(A(\omega) e^{i\omega t} \right)$$

$$A(\omega) = |A(\omega)| e^{i\alpha(\omega)}$$

$$\bar{x}(t) = 2 f_0 \operatorname{Re}\left(|A(\omega)| e^{i(\omega t + \alpha(\omega))} \right)$$

$$\boxed{\bar{x}(t) = 2 f_0 \, |A(\omega)| \cos(\omega t + \underline{\underline{\alpha(\omega)}})}$$

$$|A(\omega)| = \sqrt{A^*(\omega) \cdot A(\omega)}$$

$$= \frac{1}{2} \sqrt{\frac{1}{(-\omega^2 - i\gamma\omega + \omega_0^2)(-\omega^2 + i\gamma\omega + \omega_0^2)}}$$

$$= \frac{1}{2} \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}}$$

$z^* z = x^2 + y^2$
wenn $z = x + iy$

Resonanz

$$x(t) = \underbrace{c_1 x_1(t) + c_2 x_2(t)}_{t \gg 2/\gamma} + \underline{\underline{\bar{x}(t)}}$$

(fällt exponentiell ab)

bestimmt das Langzeitverhalten

$\alpha(\omega)$:

$$\tan \alpha(\omega) = \frac{\operatorname{Im} A(\omega)}{\operatorname{Re} A(\omega)} = \frac{-\gamma\omega}{\omega_0^2 - \omega^2}$$

wenn $z = \dfrac{1}{a+ib}$

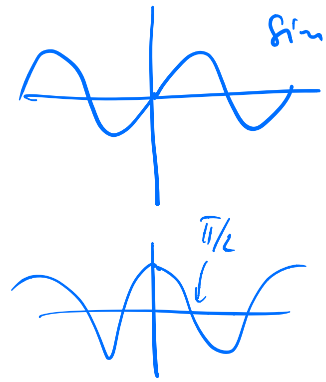
$$\Rightarrow z = \frac{a-ib}{a-ib} \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$$

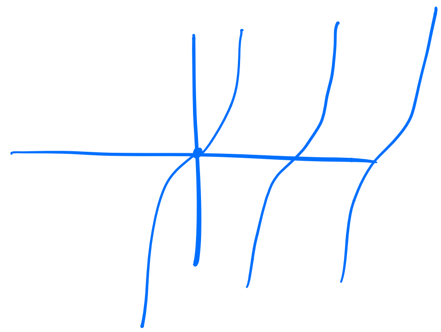
$$A = \frac{1}{2} \frac{\omega_0^2 - \omega^2 - i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2}$$

bei $\omega = \omega_0$ $\quad \alpha = \frac{\pi}{2}$

$\omega = 0$ $\quad \alpha = 0, \pi$

$\omega \to \infty$ $\quad \alpha = \pi, 0$



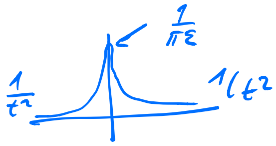


Dirac'sche Delta "Funktion"

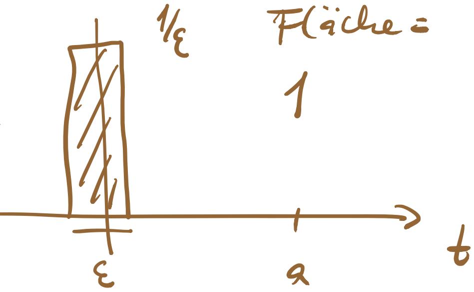
$$\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon & \text{wenn } |t| < \frac{\varepsilon}{2} \\ 0 & \text{sonst} \end{cases}$$

Lorentz funktion

$$\delta(t) = \frac{1}{\pi} \frac{\varepsilon}{\varepsilon^2 + t^2}$$



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$



Fläche = 1

$|a| > \varepsilon$

?

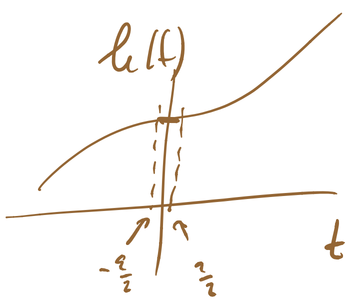
$$\boxed{\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)}$$

eigentlich kann der Grenzfall nur "unter dem Integral" durchgeführt werden.

$$\int_{-a}^{a} \delta_\varepsilon(t)\,dt = 1 \qquad |a| > \varepsilon$$

$$\underset{\varepsilon \to 0}{\Rightarrow} \qquad \int_{-a}^{a} \delta(t)\,dt = 1 \qquad |a| > 0$$

$$\lim_{\varepsilon \to 0} \int_{-a}^{a} dt\, h(t)\, \delta_\varepsilon(t) = \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} h(t)\, \delta_\varepsilon(t)\, dt$$



$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} h(t)\, dt$$

$$\simeq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\, h(0) \int_{-\varepsilon/2}^{\varepsilon/2} dt$$

$$= h(0)$$

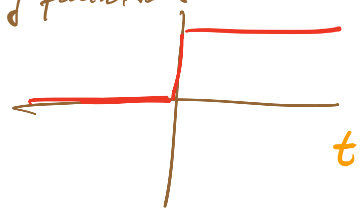
$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(t')\, \delta_\varepsilon(\underbrace{t - t'}_{t''})\, dt' = \lim\ldots \int_{-\infty}^{\infty} f(t - t'')\, \delta_\varepsilon(t'')\, dt''$$

$$= f(t)$$

$$\int_{-\infty}^{\infty} f(t')\,\delta(t-t')\,dt' = f(t) \quad !!$$

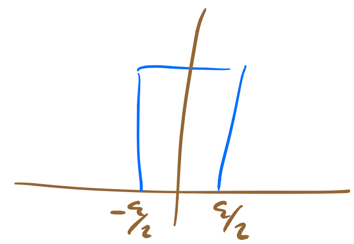
Heavyside - Sprungfunktion

$$\Theta(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



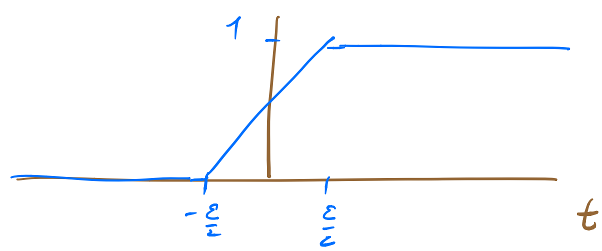
genauer

$$\Theta_\varepsilon(t) = \begin{cases} 0 & t \leq -\frac{\varepsilon}{2} \\ \frac{1}{\varepsilon}\left(t+\frac{\varepsilon}{2}\right) & -\frac{\varepsilon}{2} < t < \frac{\varepsilon}{2} \\ 1 & t \geq \frac{\varepsilon}{2} \end{cases}$$



$$\int_{-\infty}^{t} \delta_\varepsilon(t')\,dt' = \Theta_\varepsilon(t)$$

$$\Rightarrow \quad \frac{d\,\delta_\varepsilon(t)}{dt} = \Theta_\varepsilon(t)$$



da: $\lim_{\varepsilon\to 0} \Theta_\varepsilon(t) = \Theta(t)$

$$\int_{-\infty}^{t} \delta(t')\,dt' = \Theta(t) \quad \to \quad \text{"} \frac{d\Theta(t)}{dt} = \delta(t) \text{"}$$

was passiert wenn $f(t) = \delta(t-t')$?

$$\boxed{\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x(t) = \delta(t-t')}$$!!

- $\lim\limits_{\varepsilon \to 0^+} x(t'-\varepsilon) = 0$
- $\lim\limits_{\varepsilon \to 0^+} \left.\frac{dx}{dt}\right|_{t'-\varepsilon} = 0$

$$x_{t'}(t) = G(t,t')$$ Green'sche Funktion

$$= \underline{\underline{G(t-t')}}$$

$$\left(\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x(t)\right) f(t') = \delta(t-t')\, f(t')$$

$$\int_{-\infty}^{\infty} dt' \underbrace{\left(\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2\right)}\, G(t-t')\, f(t') = \int_{-\infty}^{\infty} \delta(t-t')\, f(t')\, dt'$$

$$= f(t)$$

$$\left(\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2\right) \bar{x}(t) = f(t)$$

$$\bar{x}(t) = \int_{-\infty}^{\infty} G(t-t') f(t') \, dt'$$

Man kann die spezielle Lösung jeder Inhomogenität $f(t)$ finden, wenn man nur die Lösung $G(t-t')$ findet, für die $f(t) = \delta(t-t')$!