Bestimmung der Green'schen Funktion

$$\frac{d^2x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + \omega_0^2 x(t) = \delta(t-t')$$

$$x(t<t') = 0$$

$$\left.\frac{dx}{dt}\right|_{t<t'} = 0$$

$\boxed{x(t) \text{ sei endlich und stetig}}$

$$x_{t'}(t) = G(t-t') \leftarrow$$

$$\int_{t'-\varepsilon}^{t'+\varepsilon} dt \left(\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2\right) G(t-t') = \underbrace{\int_{t'-\varepsilon}^{t'+\varepsilon} \delta(t-t')\, dt}_{1}$$

die linke Seite:

i) $$\lim_{\varepsilon\to 0} \int_{t'-\varepsilon}^{t'+\varepsilon} \omega_0^2 G(t-t')\, dt = \omega_0^2 G(0) \lim_{\varepsilon\to 0} \int_{t'-\varepsilon}^{t'+\varepsilon} dt$$

$$= \omega_0^2 G(0) \lim_{\varepsilon\to 0} 2\varepsilon = 0$$

ii) $$\lim_{\varepsilon\to 0} \gamma \int_{t'-\varepsilon}^{t'+\varepsilon} \frac{d}{dt} G(t-t')\, dt = \gamma \lim_{\varepsilon\to 0} \left[G(\varepsilon) - G(-\varepsilon)\right]$$

$$= 0$$

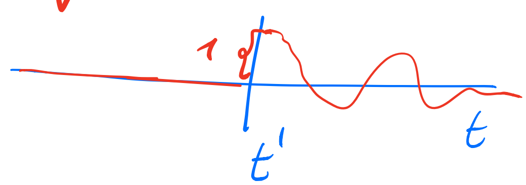
iii)

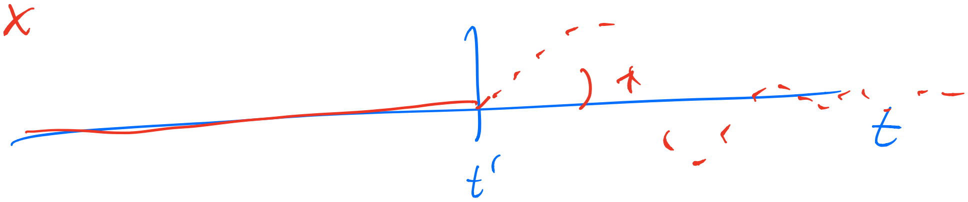
$$\lim_{\varepsilon\to 0}\int_{t'-\varepsilon}^{t'+\varepsilon} dt\, \frac{d^2 G(t-t')}{dt^2} = \lim_{\varepsilon\to 0}\left(\frac{dG(t)}{dt}\Big|_{t=\varepsilon} - \frac{dG(t)}{dt}\Big|_{t=-\varepsilon}\right)$$

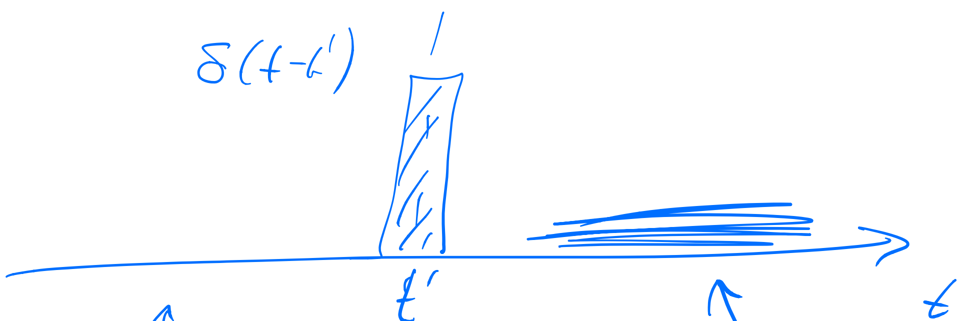
$$\Rightarrow \lim_{\varepsilon\to 0}\left(\frac{dG(t)}{dt}\Big|_{t=\varepsilon} - \underbrace{\frac{dG(t)}{dt}\Big|_{t=-\varepsilon}}_{=0}\right) = 1$$

Am Anfang in Ruhe!

$$\boxed{\frac{dG(t)}{dt}\Big|_{t=0^+} = 1}$$







wir kennen die Lösung

$G(t<t')=0, \quad \frac{dG(t<t')}{dt}=0$

wir kennen die Lösung (homogenes Problem)

$$x_{hom}(t) = e^{-\frac{\gamma}{2}t}\left(a\cos\Omega t + b\sin\Omega t\right)$$

$$\Omega = \sqrt{\omega_0^2 - \gamma^2/4}$$

$$\Longrightarrow$$

$$G(t-t') = e^{-\frac{\gamma}{2}(t-t')}\Big(a\cos[\Omega(t-t')] + b\sin[\Omega(t-t')]\Big)$$

$$G(0) = 0 \quad \Longrightarrow \quad a = 0$$

$$\left.\frac{dG(t)}{dt}\right|_{t=0} = 1 = b\,\Omega$$

$$\frac{dx_{hom}}{dt} = -\frac{\gamma}{2}e^{-\gamma/2\,t}\,b\sin(\Omega t) + b\,\Omega\,e^{-\gamma/2\,t}\cos(\Omega t)$$

$$t\to 0 \quad = b\,\Omega$$

$$\boxed{G(t-t') = \theta(t-t')\,\frac{1}{\Omega}\,e^{-\frac{\gamma}{2}(t-t')}\sin(\Omega(t-t'))}$$

$$f(t) = f_0 \cos(\omega t) = \frac{f_0}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$\bar{x}(t) = \int_{-\infty}^{\infty} dt' \, G(t-t') \, f(t')$$

$$= \frac{f_0}{2} \int_{-\infty}^{\infty} dt' \, G(t-t') \, e^{i\omega t'}$$

$$t - t' > 0$$

$$= \frac{f_0}{2} \int_{-\infty}^{t} dt' \, \frac{1}{\Omega} e^{-\frac{\gamma}{2}(t-t')} \sin(\Omega(t-t')) e^{i\omega t'}$$

$$= \frac{f_0}{2} \int_{-\infty}^{t} dt' \frac{1}{2\Omega i} e^{-\frac{\gamma}{2}(t-t')} \left(e^{i\Omega(t-t')} - e^{-i\Omega(t-t')}\right) e^{i\omega t'}$$

$$s = t' - t$$

$$= \frac{f_0}{2} \int_{-\infty}^{0} ds \, \frac{1}{2i\Omega} e^{+\frac{\gamma}{2}s} \left(e^{-i\Omega s} - e^{+i\Omega s}\right) e^{i\omega s} \, e^{i\omega t}$$

$$\int_{-\infty}^{0} ds \, e^{+as+ibs} = \frac{1}{a+ib}$$

$$= \frac{f_0}{2} \frac{1}{2i\Omega} e^{i\omega t} \left(\frac{1}{\frac{\Gamma}{2} + i(\omega - \Omega)} - \frac{1}{\frac{\Gamma}{2} + i(\omega + \Omega)} \right)$$

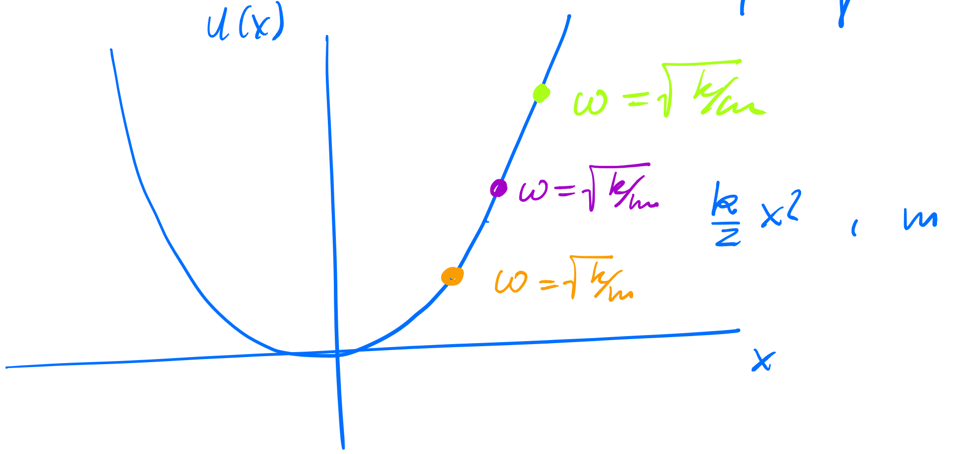
$$= \frac{f_0}{2} \frac{1}{2i\Omega} e^{i\omega t} \qquad \frac{2i\Omega}{\frac{\gamma^2}{4} + i\gamma\omega - \omega^2 + \Omega^2}$$

$$\Omega^2 = \omega_0^2 - \gamma^2/4$$

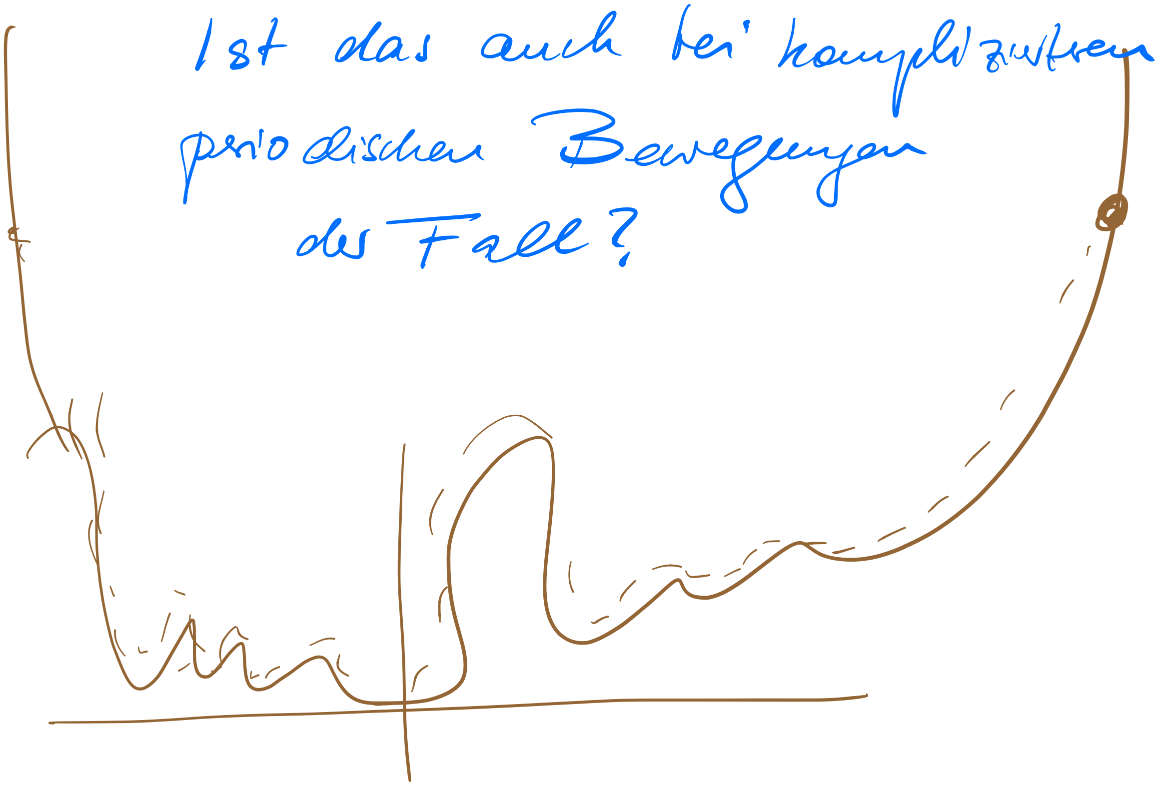
$$\bar{X}_{inh}(t) = \frac{f_0}{2} e^{i\omega t} \qquad \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

$$= f_0 \, e^{i\omega t} \quad A(\omega)$$

die Frequenz des harmonischen Oszillators ist unabh. von den Anfangsbedingungen!



Ist das auch bei komplizierteren periodischen Bewegungen der Fall?



anharmonische eindimensionale Oszillatoren

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + U(x) = E$$

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)}$$

es muß gelten, dass $U(x) \leq E$

$$\pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\left(E - U(x')\right)}} = \int_{t_0}^{t} dt' = t - t_0$$

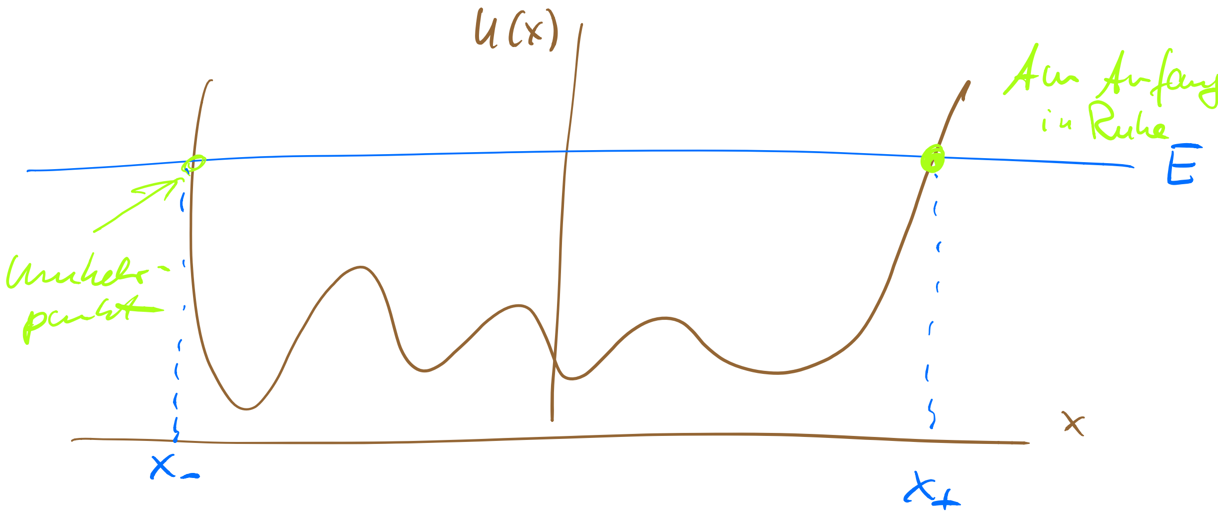
$x_0 = x(t_0)$

$x = x(t)$

$$t = t_0 \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{E - U(x')}}$$

implizite Bestimmungsgleichung f. $x(t)$!

betrachten wir ein periodisches System



periode

$$T = 2\sqrt{\frac{m}{2}} \int_{x_-}^{x_+} \frac{dx'}{\sqrt{E-U(x')}} \qquad x_- < x_+$$

$$U(x) = r\, x^n \qquad n \text{ gerade}$$

$$U(x_\pm) = E \quad \Longrightarrow \quad x_\pm = \pm \left(\frac{E}{r}\right)^{1/n}$$

$$T(E) = \sqrt{2m} \int_{x_-}^{x_+} \frac{dx'}{\sqrt{E - r x^n}} \qquad s = \frac{x}{x_+}$$

$$= \frac{E^{1/n}}{r^{1/n}} \sqrt{\frac{2m}{E}} \int_{-1}^{1} \frac{ds}{\sqrt{1-s^n}}$$

$$T(E) = \frac{\sqrt{2m}}{\gamma^{1/n}} E^{1/n - 1/2} \; C_n$$

$$C_n = \int_{-1}^{1} \frac{ds}{\sqrt{1 - s^n}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{n}\right)}{\Gamma\left(\frac{2n+1}{2n}\right)}$$

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt$$

$n=2$ $\quad C_2 = \pi$

$C_4 = 2.622\ldots$

$C_6 = 2.4285\ldots$

$T(E)$ is nur für den harmonischen Oszillator unabhängig von E!