KIT Department of Physics

Classical Theoretical Physics I

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Practice Exam / Solutions

Problem 1. Warm-up **[4 Points]**

Answer the following questions in a few keywords:

(a) [1 Point] What are the properties of orthogonal matrices? Is a rotation matrix orthogonal?
A square matrix Q is called orthogonal if its transpose is equal to its inverse: [0.5 Points]

$$Q^T = Q^{-1}$$

The determinant of an orthogonal matrix is always either $+1$ or -1:

$$\det(Q) \in \{+1, -1\}$$

Yes. A rotation matrix is the most common type of orthogonal matrix with the determinant: [0.5 Points]

$$\det(Q) = 1$$

(b) [1 Point] Through which expression do you determine the associated potential for a given potential force field $\mathbf{F}(\mathbf{r})$?

$$U(\mathbf{r}) - U(\mathbf{r}_0) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathrm{d}\mathbf{r}' \cdot \mathbf{F}(\mathbf{r}')$$

(c) [2 Points] A one-dimensional harmonic oscillator with Stokes friction is described by the equation of motion $m\ddot{x} = -r\dot{x} - kx$. What is the natural frequency of the undamped oscillator? Which qualitatively different types of motion are observed and for which friction r?

Answer:
The natural frequency is $\omega_0 = \sqrt{\frac{k}{m}}$. [0.5 Points]
The following cases must be considered separately:

$\omega_0^2 < \left(\frac{r}{2m}\right)^2$ Overdamped case (Kriechfall) [0.5 Points]

$\omega_0^2 = \left(\frac{r}{2m}\right)^2$ Critically damped case, and [0.5 Points]

$\omega_0^2 > \left(\frac{r}{2m}\right)^2$ Underdamped case (Schwingfall). [0.5 Points]

Problem 2. Conservative Force Field, Potential **[5 Points]**

Given is the force field:

$$\mathbf{F}(\mathbf{r}) = \begin{pmatrix} ax+by \\ cx+dy \\ ez \end{pmatrix}$$

(a) [1 Point] What condition must the constants $a, b, c, d, e \neq 0$ satisfy for $\mathbf{F}(\mathbf{r})$ to be a conservative force field?

We consider the condition $\nabla \times \mathbf{F}(\mathbf{r}) = 0$ and obtain: [0.25 Points]

$$\begin{pmatrix} 0 \\ 0 \\ c-b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

[0.5 Points]

For $c = b$, the force field is conservative. [0.25 Points]

(b) [1.5 Points] Determine the potential $U(\mathbf{r})$. As a check, calculate the force field resulting from the found potential.

We calculate the potential $U(\mathbf{r})$ via the line integral using the parametrization $\mathbf{r}(t) = (xt, yt, zt)$ with $t \in [0,1]$:

$$\begin{aligned} U(x,y,z) - U(0,0,0) &= -\int_0^1 \begin{pmatrix} axt+byt \\ bxt+dyt \\ ezt \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} dt \\ &= -[(ax+by)x + (bx+dy)y + ez^2] \int_0^1 t\,dt \\ &= -\frac{1}{2}(ax^2 + 2bxy + dy^2 + ez^2) \end{aligned}$$

[1 Point]

Check:

$$\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r}) = \begin{pmatrix} ax+by \\ bx+dy \\ ez \end{pmatrix}.$$

[0.5 Points]

(c) [2 Points] Determine, by explicit calculation of the line integrals, the work done on a mass point along the paths:

(i) C_1: On a direct path from $(x,y,z) = (0,0,0)$ to $(1,0,1)$.

With the parametrization $\mathbf{r}(t) = (t, 0, t)$ for $t \in [0,1]$:

$$W_1 = -\int_0^1 \begin{pmatrix} at \\ bt \\ et \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} dt = -(a+e)\int_0^1 t\,dt = -\frac{a+e}{2}$$

[0.5 Points]

(ii) C_2: In a semicircle in the xy-plane from $(1,0,1)$ via $(0,-1,1)$ to $(-1,0,1)$.

With the parametrization $\mathbf{r}(t) = (\cos(t), -\sin(t), 1)$ for $t \in [0, \pi]$: [1 Point]

$$
\begin{aligned}
W_2 &= -\int_0^{\pi} \begin{pmatrix} a\cos(t) - b\sin(t) \\ b\cos(t) - d\sin(t) \\ e \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ -\cos(t) \\ 0 \end{pmatrix} dt \\
&= \int_0^{\pi} (a\sin t\cos t - b\sin^2 t + b\cos^2 t - d\sin t\cos t) dt \\
&= \int_0^{\frac{\pi}{2}} [(a-d)\sin t\cos t + b(\cos^2 t - \sin^2 t)] dt \\
&+ \int_0^{\frac{\pi}{2}} [(a-d)\cos t(-\sin t) + b(\sin^2 t - \cos^2 t)] dt = 0
\end{aligned}
$$

(iii) C_3: On a direct path from $(-1,0,1)$ to $(0,0,0)$.

With the parametrization $\mathbf{r}(t) = (-t, 0, t)$ for $t \in [1, 0]$:

$$
W_3 = -\int_1^0 \begin{pmatrix} -at \\ bt \\ et \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} dt = (a+e)\int_0^1 t\,dt = \frac{a+e}{2}
$$

[0.5 Points]

(d) [0.5 Points] Now use the fact that the force field is conservative and verify your results from part c).

$$
\begin{aligned}
W_1 &= U(1,0,1) - U(0,0,0) = -\frac{1}{2}(a+e) \\
W_2 &= U(-1,0,1) - U(1,0,1) = 0 \\
W_3 &= U(0,0,0) - U(-1,0,1) = \frac{1}{2}(a+e)
\end{aligned}
$$

$$
W_1 + W_2 + W_3 = 0
$$

Problem 3. Particular Solution **[2 Points]**

We want to derive a general formula for calculating a particular solution of the differential equation:

$$
\ddot{x} + \omega^2 x = f(t)
$$

(a) [2 Points] Decompose the left side of this equation into linear factors:

$$
\left(\frac{\mathrm{d}}{\mathrm{d}t} + i\omega\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - i\omega\right) x(t) = f(t)
$$

and set:

$$y(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t} - i\omega\right) x(t).$$

You obtain an inhomogeneous linear first-order ODE for $y(t)$. Solve this ODE by variation of constants with the ansatz:

$$y(t) = u(t)e^{-i\omega t}.$$

Solution

(a) [2 Points] We first factor the differential operator:

$$\ddot{x} + \omega^2 x = \left(\frac{d}{dt} + i\omega\right)\left(\frac{d}{dt} - i\omega\right) x(t).$$

Define:

$$y(t) = \left(\frac{d}{dt} - i\omega\right) x(t).$$

Then it follows:

$$\left(\frac{d}{dt} + i\omega\right) y(t) = f(t),$$

which is the linear first-order differential equation:

$$\dot{y} + i\omega y = f(t).$$

[0.5 Points]

Variation of Constants We set:

$$y(t) = u(t)e^{-i\omega t}.$$

Then:

$$\dot{y} = \dot{u}e^{-i\omega t} - i\omega u e^{-i\omega t}.$$

Substituting into the differential equation:

$$\dot{u}e^{-i\omega t} - i\omega u e^{-i\omega t} + i\omega u e^{-i\omega t} = f(t).$$

[process : 0.5 Points] The terms with $i\omega$ cancel out:

$$\dot{u}e^{-i\omega t} = f(t).$$

[process: 0.25 Points]

Thus:

$$\dot{u} = f(t)e^{i\omega t}.$$

Integration yields:

$$u(t) = \int f(t')e^{i\omega t'} dt'.$$

This results in:

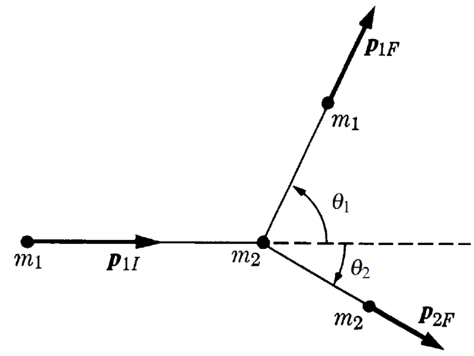
$$y(t) = e^{-i\omega t} \int f(t')e^{i\omega t'} dt'.$$

[Process and main answer 0.5 + 0.25 Points]

This is the general solution for $y(t)$.

Problem 4. Elastic Collision **[4 Points]**

Consider a "projectile" particle of mass m_1 and initial momentum p_{1I}, which collides with a "target" particle of mass m_2 that is initially at rest.
After the collision, particle m_1 is scattered by an angle θ_1 and has the final momentum p_{1F}. Particle m_2 recoils with momentum p_{2F} at an angle θ_2 relative to the original direction of p_{1I}.



(a) [1 Point] Set up the equations for the conservation of linear momentum along the direction of p_{1I} and perpendicular to it.
The conservation of momentum $\vec{p}_{before} = \vec{p}_{after}$ is decomposed into components:

- **Parallel to the incident direction (x-axis):**

$$p_{1I} = p_{1F} \cos\theta_1 + p_{2F} \cos\theta_2$$

[0.5 Points]

- **Perpendicular to the incident direction (y-axis):**

$$0 = p_{1F} \sin\theta_1 - p_{2F} \sin\theta_2 \implies p_{2F} \sin\theta_2 = p_{1F} \sin\theta_1$$

[0.5 Points]

(b) [1 Point] Use your equations from (a) to show that:

$$p_{2F}^2 = p_{1I}^2 + p_{1F}^2 - 2p_{1I}p_{1F} \cos\theta_1$$

To eliminate θ_2, we isolate the terms with θ_2 and use $\sin^2\theta + \cos^2\theta = 1$:

$$(p_{2F} \cos\theta_2)^2 = (p_{1I} - p_{1F} \cos\theta_1)^2 = p_{1I}^2 - 2p_{1I}p_{1F} \cos\theta_1 + p_{1F}^2 \cos^2\theta_1$$
$$(p_{2F} \sin\theta_2)^2 = (p_{1F} \sin\theta_1)^2 = p_{1F}^2 \sin^2\theta_1$$

[0.5 Points]

Adding both equations yields:

$$p_{2F}^2 = p_{1I}^2 + p_{1F}^2 - 2p_{1I}p_{1F} \cos\theta_1$$

[0.5 Points]

(c) [2 Points] Given the energy conservation relationship:

$$\frac{p_{1I}^2 - p_{1F}^2}{m_1} = \frac{p_{2F}^2}{m_2}.$$

Substitute the expression from (b) to derive the quadratic equation for the ratio $\frac{p_{1F}}{p_{1I}}$. Show in particular that:

$$\frac{p_{1F}}{p_{1I}} = \frac{m_1}{m_1+m_2}\cos\theta_1 \pm \left[\left(\frac{m_1}{m_1+m_2}\right)^2 \cos^2\theta_1 + \frac{m_2-m_1}{m_1+m_2}\right]^{1/2}.$$

Substituting p_{2F}^2 into the energy conservation $\frac{p_{1I}^2-p_{1F}^2}{m_1} = \frac{p_{2F}^2}{m_2}$:

$$\frac{p_{1I}^2 - p_{1F}^2}{m_1} = \frac{p_{1I}^2 + p_{1F}^2 - 2p_{1I}p_{1F}\cos\theta_1}{m_2}$$

Rearranging for p_{1F}:

$$(m_1+m_2)p_{1F}^2 - (2m_1 p_{1I}\cos\theta_1)p_{1F} + (m_1-m_2)p_{1I}^2 = 0$$

Dividing by $(m_1+m_2)p_{1I}^2$ and applying the quadratic formula for $x = \frac{p_{1F}}{p_{1I}}$: [process 1.75 Points]

$$\frac{p_{1F}}{p_{1I}} = \frac{m_1}{m_1+m_2}\cos\theta_1 \pm \left[\left(\frac{m_1}{m_1+m_2}\right)^2 \cos^2\theta_1 + \frac{m_2-m_1}{m_1+m_2}\right]^{1/2}$$

[0.25 Points]