Aufgabe 5

a) Partialbruchzerlegung

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{(1-x)}$$

$$1 = A(1-x) + Bx$$

$$1 + 0x = A - Ax + Bx$$

$$\underline{\underline{1 = A}}$$

$$1 + 0x = 1 - x + Bx$$

$$x = Bx$$

$$\underline{\underline{1 = B}}$$

b) DGL lösen

$$\dot v(t) = a v(t)\,(1 - v(t))$$

$$\frac{\dot v(t)}{v(t)} = a\,(1 - v(t)) \qquad ; \quad v(t) = v$$

$$\frac{\dot v}{v} \cdot \frac{1}{1-v} = a$$

$$\Leftrightarrow \quad \frac{\frac{dv}{dt}}{v} \cdot \frac{1}{1-v} = a$$

$$\Leftrightarrow \quad \frac{1}{v} \cdot \frac{1}{1-v}\, dv = a\, dt'$$

$$\Leftrightarrow \quad \frac{1}{v(1-v)}\, dv = a\, dt'$$

siehe a)

$$\Leftrightarrow \quad \left(\frac{1}{v} + \frac{1}{1-v}\right) dv = a\, dt' \qquad \Big| \int_{t_0}^{t}$$

$$\Leftrightarrow \quad \ln|v| \quad \ln|1-v| \Big|^{t} = at' \Big|_{t_0}^{t}$$

$$\ln\left|\frac{v}{1-v}\right| \Big|_{t_0}^{t\,t_0} = a\,(t - t_0)$$

$$\Leftrightarrow \quad \ln\left|\frac{v(t)}{1-v(t)}\right| - \ln\left|\frac{v(t_0)}{1-v(t_0)}\right| = a(t-t_0)$$

$$\Leftrightarrow \quad \ln\left|\frac{v(t)\,(1-v(t_0))}{(1-v(t))\,v(t_0)}\right| = a(t-t_0) \qquad \Big|\, e^{\cdot}$$

$$\Leftrightarrow \quad \frac{v(t)\,(1-v(t_0))}{1-v(t))\,v(t_0} = e^{a(t-t_0)}$$

$$\Leftrightarrow \quad \frac{v(t)}{1-v(t)} = e^{a(t-t_0)} \cdot \frac{v(t_0)}{1-v(t_0)}$$

$$\Leftrightarrow \quad \frac{1-v(t)}{v(t)} = \frac{1}{e^{a(t-t_0)} \cdot \frac{v(t_0)}{1-v(t_0)}}$$

$$\Leftrightarrow \quad \frac{1}{v(t)} - 1 = \frac{1}{e^{a(t-t_0)}} \cdot \frac{1-v(t_0)}{v(t_0)}$$

$$\Leftrightarrow \quad \frac{1}{v(t)} - 1 = \frac{1}{e^{a(t-t_0)}} \cdot \frac{1-v(t_0)}{v(t_0)}$$

$$\Leftrightarrow \quad \frac{1}{v(t)} = \frac{1}{e^{a(t-t_0)}} \cdot \frac{1-v(t_0)}{v(t_0)} + 1$$

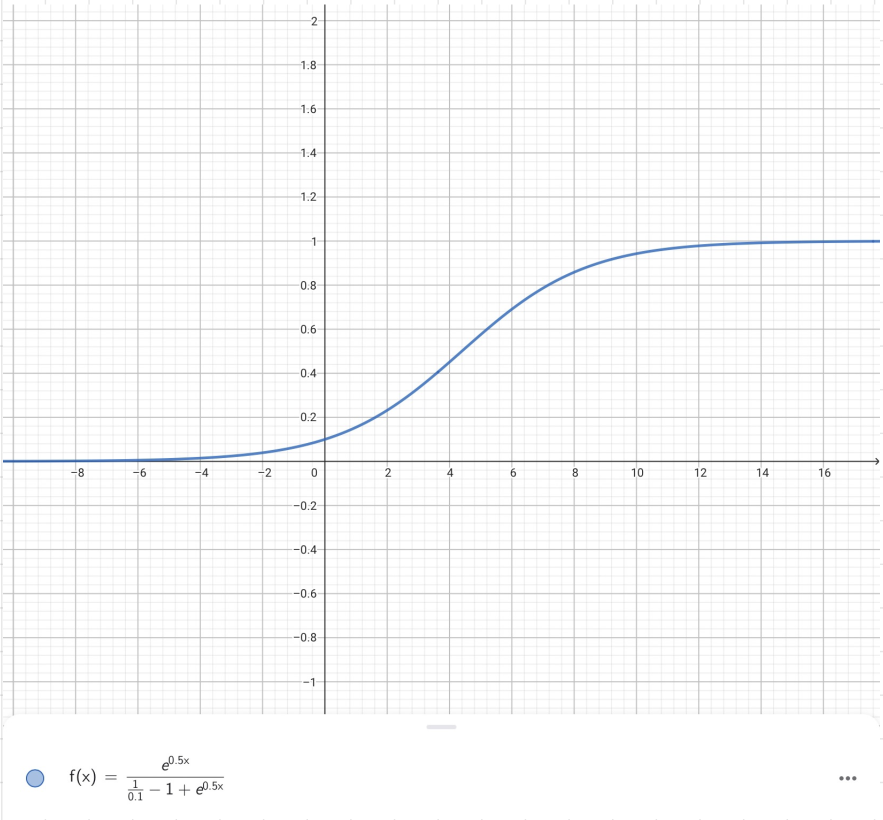
$$\Leftrightarrow \quad \frac{1}{v(t)} = \frac{1 - v(t_0) + e^{a(t-t_0)}\,v(t_0)}{e^{a(t-t_0)}\,v(t_0)}$$

$$\Leftrightarrow \quad v(t) = \frac{e^{a(t-t_0)}\,v(t_0)}{1 - v(t_0) + e^{a(t-t_0)}\,v(t_0)}$$

$$\Leftrightarrow \quad v(t) = \frac{e^{a(t-t_0)}}{\frac{1}{v(t_0)} - 1 + e^{a(t-t_0)}}$$

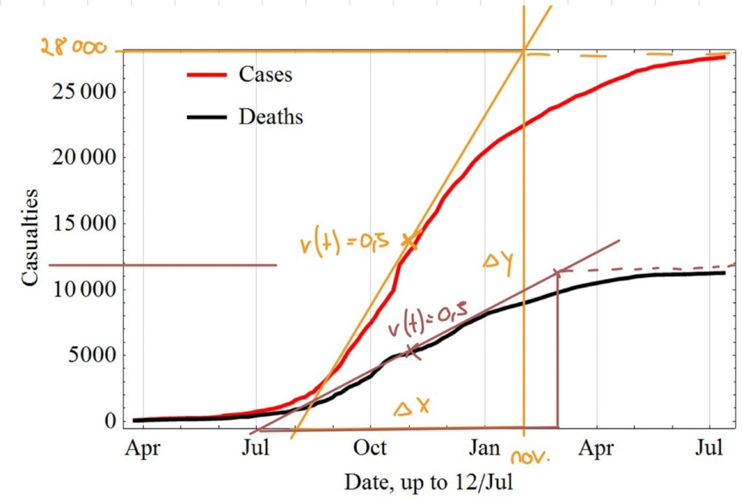
c) Vergleich Krankheitsverlauf

eine mögliche Kurve aus b):



Die Kurven haben einen ähnlichen Verlauf. Für $v(t)$ gilt aber $0 < v(t) < 1$, da $v(t)$ den Anteil der Erkrankten beschreibt.

Bestimmung von α_{rot} und $\alpha_{schwarz}$



$28\,000 \mathrel{\hat=} 1$

$\frac{\Delta y}{\Delta x} = \frac{1}{6}$

$11\,000 \mathrel{\hat=} 1$

$\frac{\Delta y}{\Delta x} = \frac{1}{8}$

rote Kurve: $v(\text{november}) = 0,5$

$\dot{v}(\text{nov.}) = \frac{1}{6}$ (t in Monaten) (siehe Steigungs-dreieck)

$$\frac{1}{6} = \alpha_r \cdot 0,5\,(1-0,5)$$

$$\alpha_r = \frac{2}{3} = 0,\overline{66}$$

schwarze Kurve: $v(\text{nov.}) = 0,5$

$\dot{v}(\text{nov.}) = \frac{1}{8}$

$$\frac{1}{8} = \alpha_s \cdot 0,5\,(1-0,5)$$

$$\alpha_s = \frac{1}{2} = 0,5$$

d) Neue Funktion

$$\dot{v}(t) = \left(\int_{t-t_{inf}}^{t} g(s)\,ds - b \right) v(t)$$

$$\frac{\frac{dv}{dt'}}{v} = G(t') - b$$

$$\frac{1}{v}\,dv = \left(G(t') - b\right)\,dt' \qquad \Big| \int_{t_0}^{t}$$

$$\ln|v|\,\Big|_{v(t_0)}^{v(t)} = \int_{t_0}^{t} G(t')\,dt' - \left(bt - bt_0\right)$$

$$\ln\left|\frac{v(t)}{v(t_0)}\right| = \int_{t_0}^{t} G(t')\,dt' - b\,(t - t_0) \qquad \big| e^{(\,)}$$

$$\frac{v(t)}{v(t_0)} = e^{\int_{t_0}^{t} G(t') - b\,dt'} \qquad \big| \cdot v(t_0)$$

$$\underline{\underline{v(t) = v(t_0)\, e^{\int_{t_0}^{t} G(t) - b(t-t_0)\,dt}}}$$

e) Simulation

$$G(t) := \int_{t-t_{inf}}^{t} g(s)\,ds$$

1. Fall: $t < 0$

$$G_1(t) = g_1 t \,\Big|_{t-t_{inf}}^{t}$$

$$G_1(t) = g_1 t - g_1 t + t_{inf}\, g_1$$

$$\underline{G_1(t) = t_{inf}\, g_1}$$

$$v(t) = v(t_0)\, e^{\int (G(t) - b)\,dt}$$

$$\underline{\underline{v(t) = v(t_0)\, e^{t_{inf} \cdot g_1 t - bt + C_1}}}$$

$$G_1(0) = t_{inf.}\, g_1 = G_1(t_{inf.})$$

2. Fall : $0 \le t \le t_{inf}$

$$G_2(t) = \int_{t-t_{inf}}^{t} g(t')\, dt'$$

$$G_2(t) = g_2 t \Big|_0^t + g_1 t \Big|_{t-t_{inf}}^0$$

$$G_2(t) = g_2 t + g_1 (t - t_{inf})$$

$$v_2(t) = v(t_0)\, e^{\int (G(t') - b)\, dt'}$$

$$v_2(t) = v(t_0)\, e^{\frac{1}{2} g_2 t^2 - \frac{1}{2} g_1 t^2 + g_1 t_{inf.} t + C_2}$$

$$G_2(0) = -g_1 (t_{inf.})$$

$$G_2(t_{inf.}) = g_2 (t_{inf.})$$

3. Fall : $t > t_{inf.}$

$$G_3(t) = \int_{t-t_{inf}}^{t} g(t')\, dt'$$

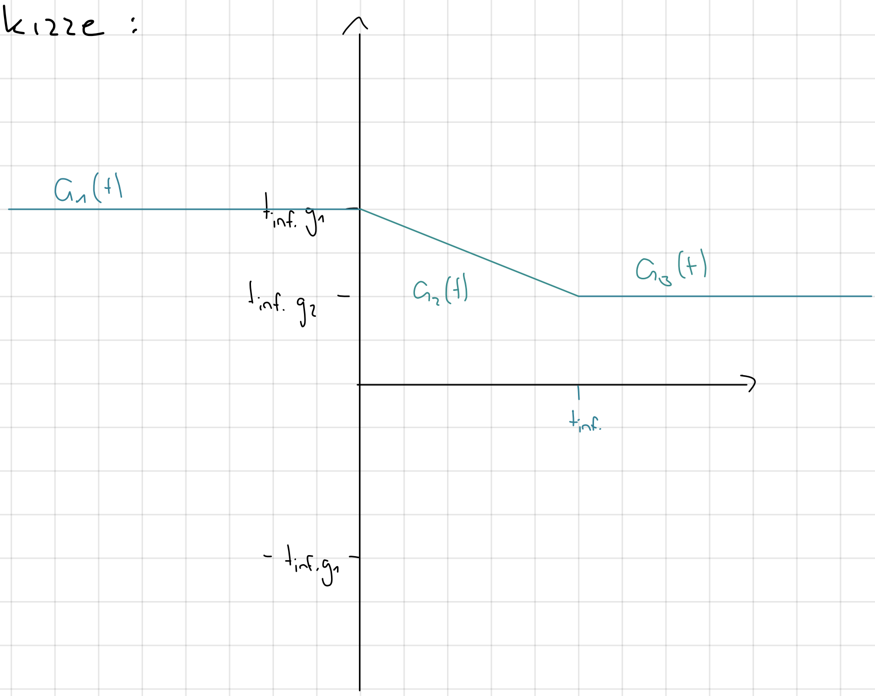
$$G_3(t) = t_{inf.}\, g_2$$

$$v_3(t) = v(t_0)\, e^{\int (G(t) - b) dt}$$

$$v_3(t) = v(t_0)\, e^{t_{inf.} g_2 t - bt + C_3}$$

$$G_3(0) = t_{inf.}\, g_2 = G_3(t_{inf.})$$

Skizze :



aus Skizze folgt :

$G_1(t)$ (also $G(t)$ für $t \leq 0$)

braucht Integrationskonstante $-2\, t_{inf.}\, g_1$

Warum hat $v(t)$ keine Knicke ?

$$v(t) = v(t_0)\, e^{\int_{t_0}^{t} G(t) - b(t - t_0)}$$

$$\dot{v}(t) = v(t) \left(G_i(t) - b \right)$$

Die Ableitung ist stetig, somit kann $v(t)$ keine Knicke haben.

f) Betrachtung der Funktion

Bestimmung der Konstanten:

$$v_1(0) = v_2(0) \Leftrightarrow C_1 = C_2 = 0$$

$$v_2(t_{inf.}) = v_3(t_{inf.})$$

$$\frac{1}{2} g_2 t^2 - \frac{1}{2} g_1 t^2 + g_1 t_{inf.} t - bt = t_{inf.} g_2 t - bt + C_3$$

$$C_3 = -\frac{1}{2} t_{inf.}^2 g_2 + \frac{1}{2} g_1 t_{inf.}^2$$

$$\underline{\underline{C_3 = \frac{1}{2} t_{inf.}^2 (g_1 - g_2)}}$$

$$v_1(t) = v_0 e^{t_{inf.} g_1 t + C_1 - bt}$$

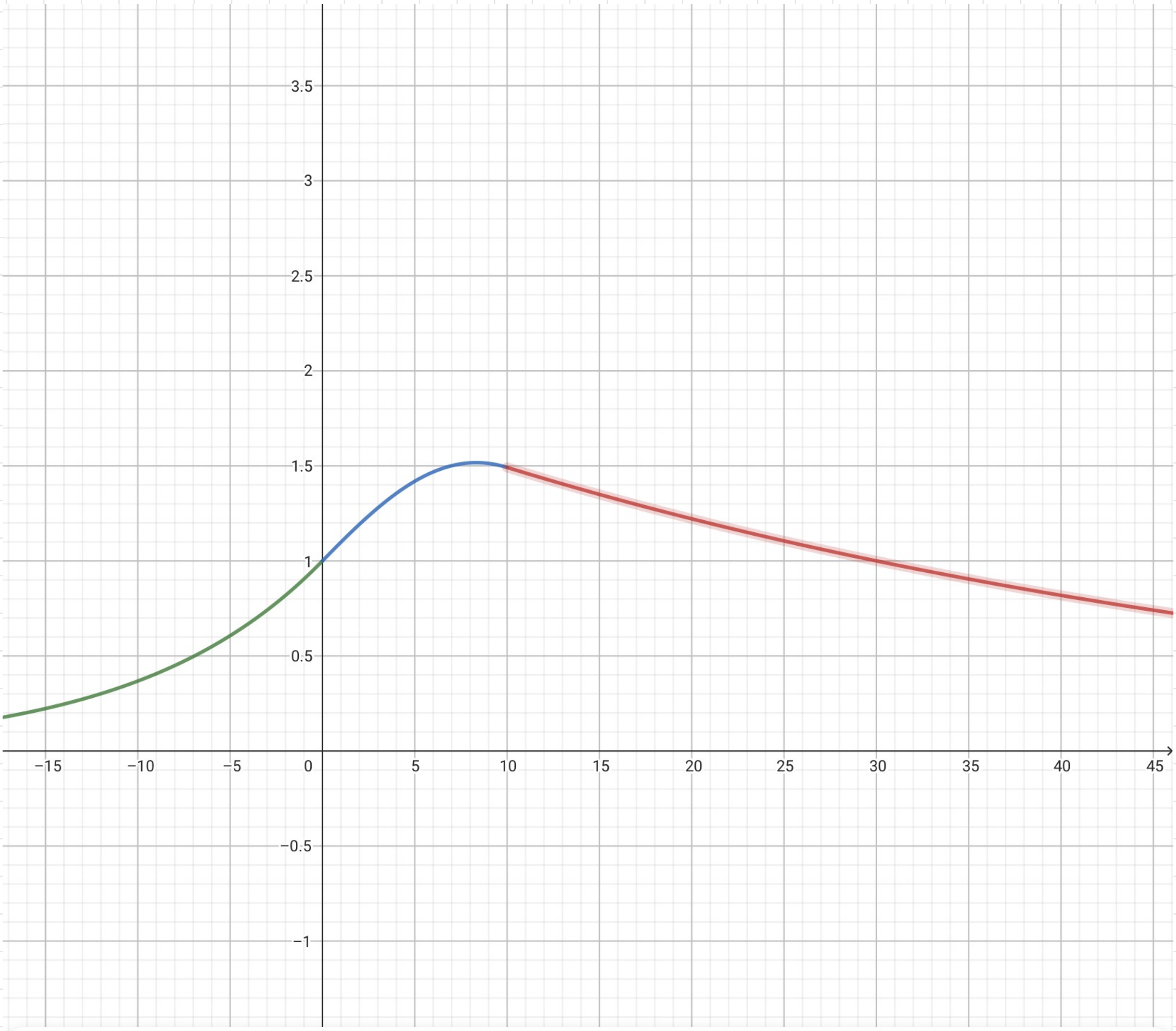
$$v_1(t) = v_0 e^{0{,}1 \frac{1}{2} t}$$

$$v_2(t) = v(t_0) e^{\frac{1}{2} g_2 t^2 - \frac{1}{2} g_1 t^2 - g_1 t_{inf.} t + C_2 - bt}$$

$$v_2(t) = v_0 e^{-0{,}006 t^2 + 0{,}1 t}$$

$$v_3(t) = v(t_0) e^{t_{inf.} g_2 t - bt + C_3}$$

$$v_3(t) = v_0 e^{-0{,}02 t + 0{,}6}$$



$f(x) = e^{0.1x}, \quad (x < 0)$

$g(x) = e^{-0.006x^2+0.1x}, \quad (0 < x < 10)$

$h(x) = e^{-0.02x+0.6}, \quad (10 < x)$

Zeitpunkt $t_{max.}$:

$$v_2'(t) = \underbrace{e^{-0{,}006t^2 + 0{,}1t}}_{\neq 0} \cdot (-0{,}012t + 0{,}1) = 0$$

$$0 = -0{,}012t + 0{,}1$$

$$0{,}1 = 0{,}012t$$

$$t_{max.} = 8{,}33$$

, siehe Skizze: Maximum
kein Minimum

$$\dot{v} = -\alpha v - \beta v^2 \quad ;$$

a) Partialbruchzerlegung

$$\frac{1}{\alpha v + \beta v^2} = \frac{1}{v(\alpha + \beta v)} = \frac{A}{\alpha + \beta v} + \frac{B}{v}$$

$$1 = Av + B(\alpha + \beta v)$$

$$1 = Av + B\alpha + B\beta v$$

$$1 + 0v = B\alpha + Av + B\beta v$$

$$1 = B\alpha$$

$$B = \frac{1}{\alpha}$$

$$1 = 1 + Av + \frac{\beta}{\alpha} v$$

$$0 = Av + \frac{\beta}{\alpha} v$$

$$A = -\frac{\beta}{\alpha}$$

$$\Rightarrow \quad \frac{1}{\alpha v + \beta v^2} = -\frac{\beta}{\alpha^2 + \beta \alpha v} + \frac{1}{\alpha v} = \frac{1}{\alpha}\left(\frac{\beta}{\alpha + \beta v} + \frac{1}{v}\right)$$

b) DGL lösen

$$\dot{v} = -\alpha v - \beta v^2 \quad ; \quad \begin{matrix} v > 0 \\ v \neq -\frac{\alpha}{\beta} \end{matrix}$$

$$\Leftrightarrow \frac{\frac{dv}{dt}}{-\alpha v - \beta v^2} = 1$$

$$\Leftrightarrow \frac{1}{-\alpha v - \beta v^2} dv = 1 dt'$$

$$\frac{1}{-\alpha v - \beta v^2} = \frac{1}{-v(\alpha + \beta v)} = \frac{A}{v} - \frac{B}{\alpha + \beta v}$$

$$\left.\begin{matrix} A = -\frac{1}{\alpha} \\ B = -\frac{\beta}{\alpha} \end{matrix}\right\} \text{analog zu a)}$$

$$\frac{1}{-\alpha v - \beta v^2} = -\frac{1}{\alpha v} + \frac{\beta}{\alpha(\alpha + \beta v)}$$

$$\frac{1}{-\alpha v - \beta v^2} dv = 1 dt'$$

$$\Leftrightarrow \left(\frac{\beta}{\alpha(\alpha + \beta v)} - \frac{1}{\alpha v} \right) dv = 1 dt'$$

$$\Leftrightarrow \frac{1}{\alpha} \left(\frac{\beta}{\alpha + \beta v} - \frac{1}{v} \right) dv = 1 dt' \quad \Big| \int_{t_0}^{t}$$

$$\Rightarrow \ln|\alpha + \beta v| - \ln|v| \Big|_{v(t_0)}^{v(t)} = \alpha (t - t_0)$$

$$\Rightarrow \ln \left| \frac{\alpha + \beta v}{v} \right| \Big|_{v(t_0)}^{v(t)} = \alpha (t - t_0)$$

$$\Leftrightarrow \ln \left| \frac{(\alpha + \beta v(t))\, v(t_0)}{v(t)\,(\alpha + \beta v(t_0))} \right| = \alpha (t - t_0) \quad \Big| \, e^{(\,)}$$

$$\Leftrightarrow \quad \frac{(\alpha + \beta v(t))\, v(t_0)}{v(t)\,(\alpha + \beta v(t_0))} = e^{\alpha(t-t_0)}$$

$$\Leftrightarrow \quad \frac{\alpha}{v(t)} + \frac{\beta v(t)}{v(t)} = \frac{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0))}{v(t_0)}$$

$$\Leftrightarrow \quad \frac{\alpha}{v(t)} + \beta = \frac{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0))}{v(t_0)}$$

$$\Leftrightarrow \quad \frac{\alpha}{v(t)} = \frac{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0))}{v(t_0)} - \beta$$

$$\Leftrightarrow \quad \alpha = \left[\frac{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0))}{v(t_0)} - \beta\right] v(t)$$

$$\Leftrightarrow \quad \frac{\alpha}{\frac{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0))}{v(t_0)} - \beta} = v(t)$$

$$\Leftrightarrow \quad v(t) = \frac{\alpha\, v(t_0)}{e^{\alpha(t-t_0)}\,(\alpha + \beta v(t_0)) - \beta v(t_0)}$$

$$\dot{v} = -\alpha v - \beta v^2$$

1. Fall: $\beta = 0$

$$\dot{v} = -\alpha v$$

$$\frac{1}{v} dv = -\alpha \, dt' \qquad \Big| \int_{t_0}^{t}$$

$$\ln\left|\frac{v(t)}{v(t_0)}\right| = -\alpha t + \alpha t_0$$

$$\underline{\underline{v_\alpha(t) = e^{\alpha(t_0 - t)} \cdot v(t_0)}}$$

2. Fall: $\alpha = 0$

$$\dot{v} = -\beta v^2$$

$$\frac{\frac{dv}{dt'}}{v^2} = -\beta$$

$$\frac{1}{v^2} dv = -\beta \, dt' \qquad \Big| \int_{t_0}^{t}$$

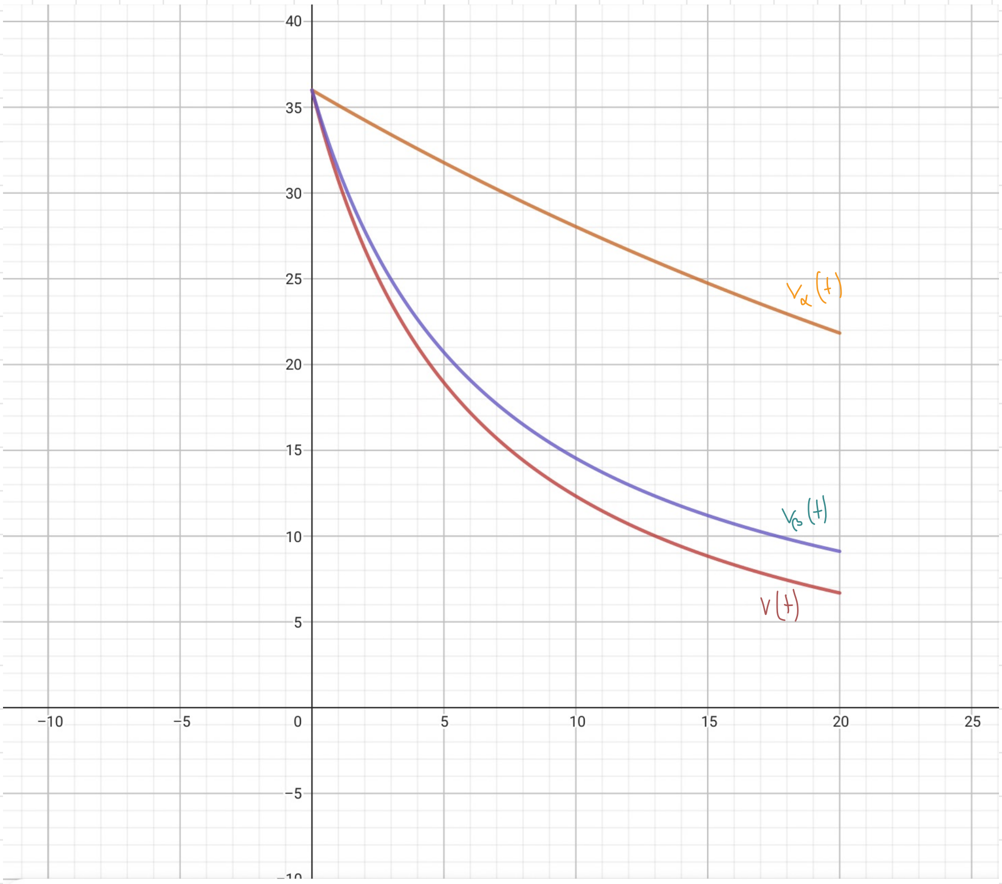
$$-\frac{1}{v}\Big|_{v(t_0)}^{v(t)} = -\beta t' \Big|_{t_0}^{t}$$

$$\beta(t - t_0) = \frac{1}{v(t)} - \frac{1}{v(t_0)}$$

$$\frac{1}{v(t)} = \beta(t - t_0) + \frac{1}{v(t_0)} \qquad \Big| \cdot v(t)$$

$$1 = v(t)\left[\beta(t - t_0) + \frac{1}{v(t_0)}\right]$$

$$\underline{\underline{v_\beta(t) = \frac{1}{\beta(t - t_0) + \frac{1}{v(t_0)}}}}$$



$$g(x) = \frac{2.5\cdot 10^{-2}\cdot 36}{e^{2.5\cdot 10^{-2}x}\left(2.5\cdot 10^{-2} + 4\cdot 10^{-3}\cdot 36\right) - 4\cdot 10^{-3}\cdot 36}, \quad (0 < x < 20)$$

$$h(x) = 36\, e^{-2.5\cdot 10^{-2}x}, \quad (0 < x < 20)$$

$$p(x) = \frac{1}{4.1\cdot 10^{-3}\,x + \frac{1}{36}}, \quad (0 < x < 20)$$

c) Wegfunktion

$$v(t) = \frac{\alpha\, v(t_0)}{e^{\alpha(t-t_0)}\,(\alpha+\beta v(t_0))-(\beta v(t_0))} \quad ; \; t_0 = 0 \text{ festgelegt}$$

$$v(t) = \alpha v(0) \cdot \frac{1}{e^{\alpha(t)}\,(\alpha+\beta v(0))-(\beta v(0))} \quad \Big| \int_0^t dt$$

$$\int_0^t v(t) = \alpha v(0) \cdot \int_0^t \frac{1}{e^{\alpha t'}(\alpha+\beta v(0))-\beta v(0)}\, dt'$$

Subst.: $u = e^{-\alpha t} \qquad \frac{du}{dt} = -\alpha e^{-\alpha t}$

$$x(t) - x_0 = \alpha v(0) \cdot \int_1^{e^{-\alpha t}} \frac{e^{-\alpha t'}}{(\alpha+\beta v(0)) - \beta v(0) u} \cdot \left(-\frac{e^{\alpha t'}}{\alpha}\right) du$$

$$x(t) - x_0 = v(0) \cdot \int_1^{e^{-\alpha t}} \frac{1}{(\alpha+\beta v(0)) - \beta v(0) u} \cdot du$$

2. Subst.: $z = (\alpha + \beta v(0)) - \beta v(0) u$

$$\frac{dz}{du} = \beta v(0)$$

$$du = \frac{dz}{\beta v(0)}$$

$$= v(0) \cdot \int_\alpha^{\alpha+\beta v(0)-\beta v(0) e^{-\alpha t}} \frac{1}{\beta v(0)\, z}\, dz$$

$$x(t) - x_0 = v(0) \left[\ln|\beta v(0) z| \frac{1}{\beta v(0)} \right]_\alpha^{\alpha+\beta v(0)-\beta v(0)e^{-\alpha t}}$$

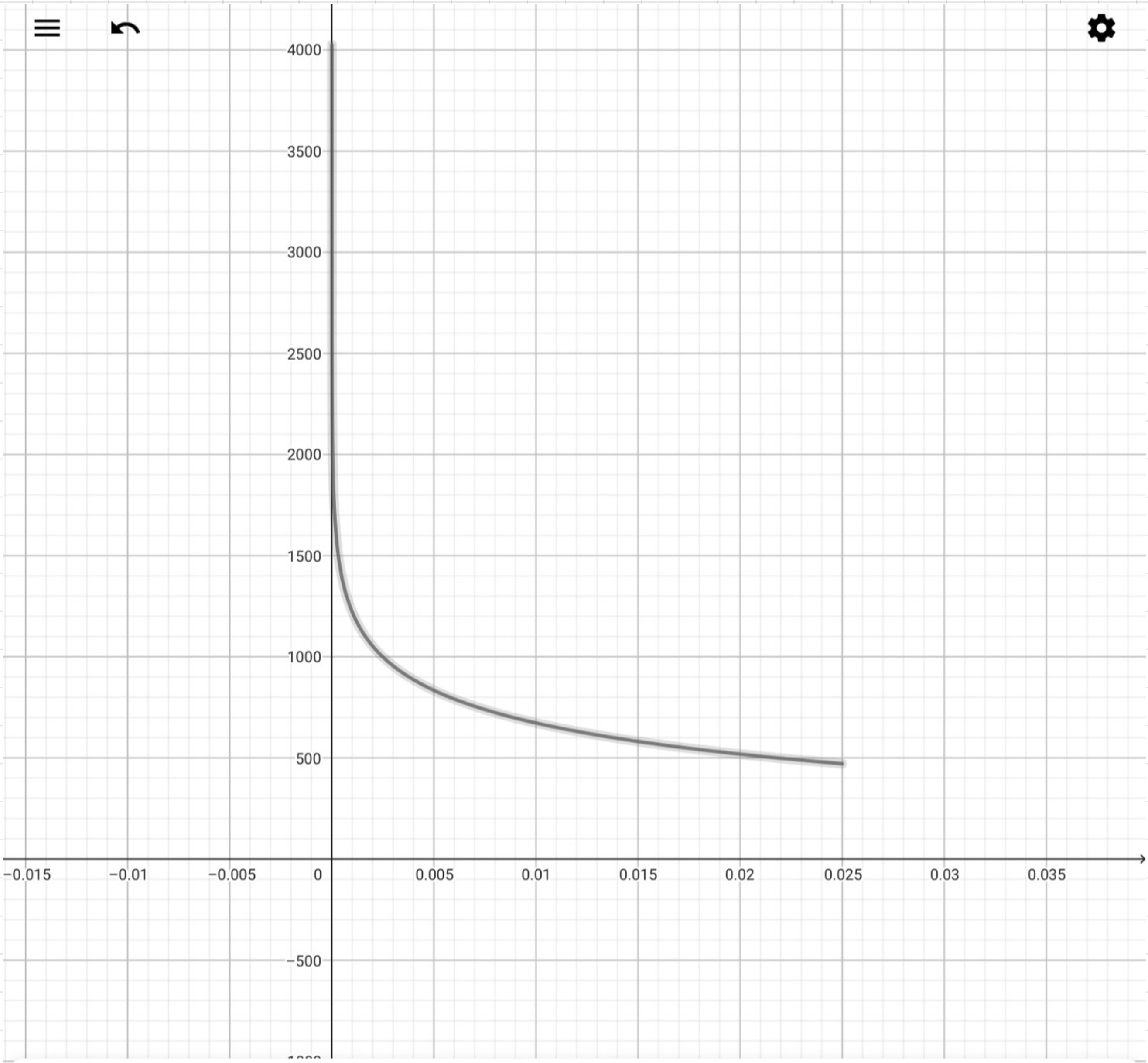
Rücksubst.:

$$x(t) - x_0 = \frac{1}{\beta} \ln \left| \frac{\beta v(0)(\alpha+\beta v(0)-\beta v(0)e^{-\alpha t})}{\beta v(0)\,\alpha} \right|$$

$$\underline{\underline{x(t) = \frac{1}{\beta} \ln \left| 1 + \frac{\beta v(0)}{\alpha} - \frac{1}{\alpha e^{\alpha t}} \right|}}$$

$$\lim_{t \to \infty} \left(\frac{1}{\beta} \ln \left| 1 + \frac{\beta v(0)}{\alpha} - \frac{1}{\alpha e^{\alpha t}} \right| \right)$$

$$= \frac{1}{\beta} \ln \left| 1 + \frac{\beta v(0)}{\alpha} \right|$$



$f(x) = \frac{1}{4.1 \cdot 10^{-3}} \ln\left(1 + \frac{4.1 \cdot 10^{-3} \cdot 36}{x}\right), \quad (0 < x < 2.5 \cdot 10^{-2})$