Aufgabe 11

$$\vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} ct \sin(\omega t) \\ ct \cos(\omega t) \end{pmatrix}$$

a) $\vec{v}(t)$

$$\vec{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} c\sin(\omega t) + \omega ct\cos(\omega t) \\ c\cos(\omega t) - \omega ct \sin(\omega t) \end{pmatrix}$$

b) $\vec{a}(t)$

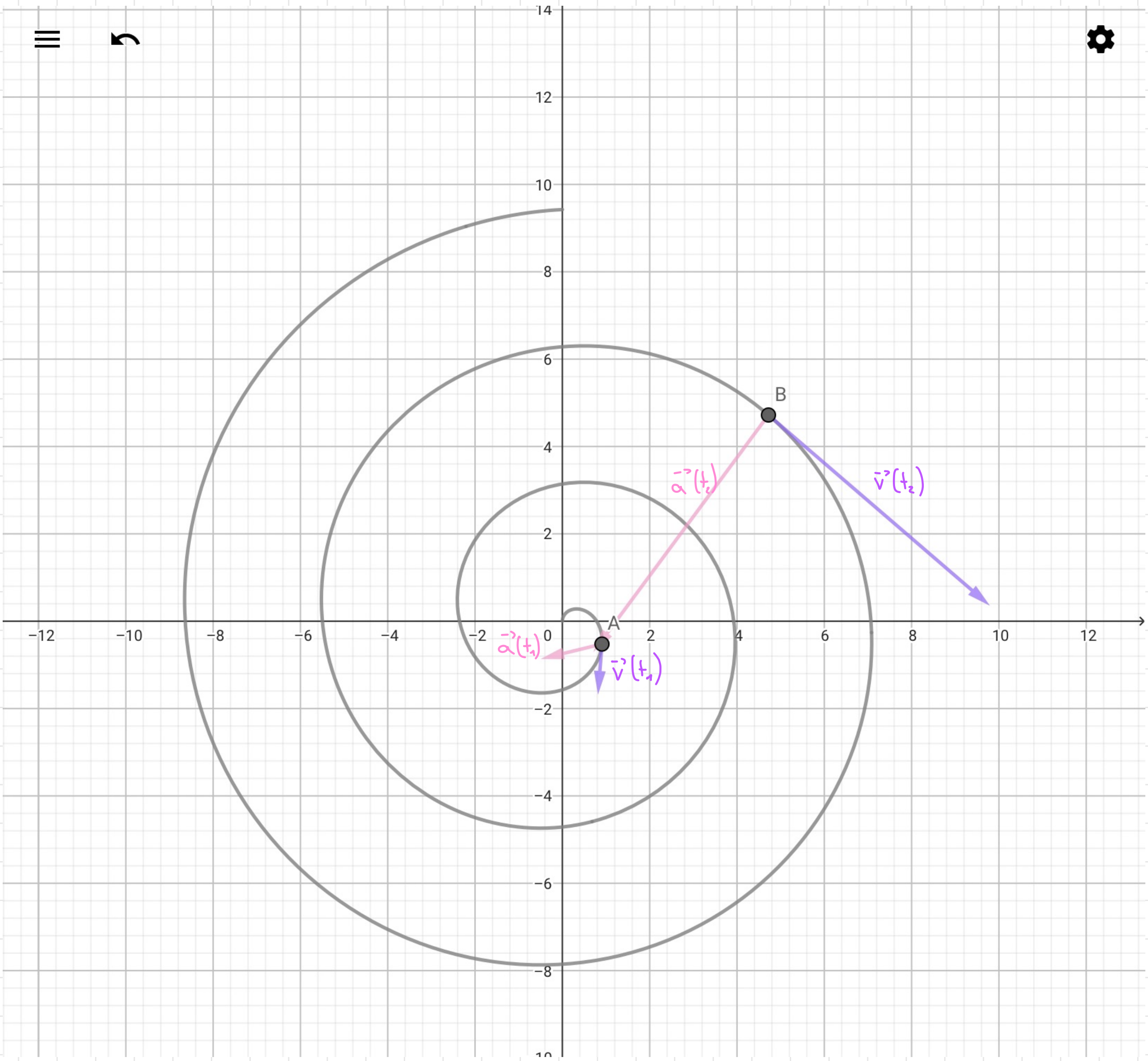
$$\vec{a}(t) = \dot{\vec{v}}(t) = \begin{pmatrix} 2\omega c\cos(\omega t) - \omega^2 ct \sin(\omega t) \\ -2\omega c\sin(\omega t) - \omega^2 ct \cos(\omega t) \end{pmatrix}$$

c) In Geogebra, für $c = \frac{\omega}{2}$, $\omega = 1$

$A = r(t_1)$; $B = r(t_2)$

Geschwindigkeit

Beschleunigung



d) (1 Punkt) Beweisen Sie

$$\int dx\,\sqrt{ax^2+2bx+c} = \frac{ac-b^2}{2a^{3/2}}\operatorname{arsinh}\frac{ax+b}{\sqrt{ac-b^2}} + \frac{ax+b}{2a}\sqrt{ax^2+2bx+c} + C,$$

falls $a > 0$ und $ac - b^2 > 0$.

d)

$$\int dx\,\sqrt{ax^2+2bx+c} = \frac{ac-b^2}{2\sqrt{a^3}} \cdot \operatorname{arsinh}\left(\frac{ax+b}{\sqrt{ac-b^2}}\right) + \frac{ax+b}{2a}\sqrt{ax^2+2bx+c} + C$$

$$\frac{d}{dx}\left[\frac{ac-b^2}{2\sqrt{a^3}} \cdot \operatorname{arsinh}\left(\frac{ax+b}{\sqrt{ac-b^2}}\right) + \frac{ax+b}{2a}\sqrt{ax^2+2bx+c}\right]$$

$$= \sqrt{ax^2+2bx+c}$$

1. $$\frac{d}{dx}\left[\frac{ac-b^2}{2\sqrt{a^3}} \cdot \operatorname{arsinh}\left(\frac{ax+b}{\sqrt{ac-b^2}}\right)\right]$$

$$= \frac{ac-b^2}{2\sqrt{a^3}} \cdot \frac{1}{\sqrt{\frac{(ax+b)^2}{ac-b^2}+1}} \cdot \frac{a}{\sqrt{ac-b^2}}$$

$$= \frac{ac-b^2}{2\sqrt{a^3}} \cdot \frac{a}{\sqrt{(ax+b)^2+ac-b^2}}$$

$$(ax+b)^2 = a^2x^2+2bax+b^2$$

$$= \frac{ac-b^2}{2\sqrt{a^3}} \cdot \frac{a}{\sqrt{a(ax^2+2bx+c)}}$$

$$= \frac{ac-b^2}{2\,a^2} \cdot \frac{a}{\sqrt{ax^2+2bx+c}}$$

$$= \frac{ac-b^2}{2a\sqrt{ax^2+2bx+c}}$$

2. $$\frac{d}{dx}\left(\frac{ax+b}{2a}\sqrt{ax^2+2bx+c}\right)$$

$$= \frac{d}{dx}\left(\left(\frac{1}{2}x+\frac{b}{2a}\right)\cdot\sqrt{ax^2+2bx+c}\right)$$

$$= \frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{ax+b}{2a}\quad\frac{ax+b}{\sqrt{ax^2+2bx+c}}$$

$$\frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{ax+b}{2a}\cdot\frac{ax+b}{\sqrt{ax^2+2bx+c}} + \frac{ac-b^2}{2a\sqrt{ax^2+2bx+c}}$$

$$= \frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{(ax+b)^2+ac-b^2}{2a\sqrt{ax^2+2bx+c}}$$

$$= \frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{a^2x^2+2bax-b^2+ac-b^2}{2a\sqrt{ax^2+2bx+c}}$$

$$= \frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{ax^2+2bx+c}{2\sqrt{ax^2+2bx+c}}$$

$$\sqrt{ax^2+2bx+c} = \frac{1}{2}\sqrt{ax^2+2bx+c} + \frac{ax^2+2bx+c}{2\sqrt{ax^2+2bx+c}}$$

$$ax^2+2bx+c = \frac{1}{2}(ax^2+2bx+c) + \frac{1}{2}(ax^2+2bx+c)$$

$$\underline{\underline{ax^2+2bx+c = ax^2+2bx+c}}$$

falls $a > 0$ und $ac - b^2 > 0$.

e) (1 Punkt) Bestimmen Sie den vom Massenpunkt zwischen den Zeitpunkten $t = 0$ und $t = T$ zurückgelegten Weg $\int_0^T dt\, |\vec{v}(t)|$ (also die Bogenlänge der Spirale).

$$\vec{v}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} c\sin(\omega t) + \omega ct\cos(\omega t) \\ c\cos(\omega t) - \omega ct\sin(\omega t) \end{pmatrix}$$

$$s = \int_0^T dt\, |\vec{v}|$$

$$|\vec{v}|^2 = \left(c\sin\omega t + \omega ct\cos(\omega t)\right)^2 + \left(c\cos(\omega t) - \omega ct\sin(\omega t)\right)^2$$

$$= c^2\sin^2(\omega t) + \omega^2c^2t^2\cos^2(\omega t) + 2\omega tc^2\sin(\omega t)\cos(\omega t)$$
$$c^2\cos^2(\omega t) + \omega^2c^2t^2\sin^2(\omega t) - 2\omega tc^2\cos(\omega t)\sin(\omega t)$$

$$= c^2\sin^2(\omega t) + c^2\cos^2(\omega t) + \omega^2c^2t^2\cos^2(\omega t)\ \omega^2c^2t^2\sin^2(\omega t)$$

$$= c^2 + \omega^2c^2t^2$$

$$|\vec{v}| = c\sqrt{(1 + \omega^2t^2)}$$

$$\int_0^T c \sqrt{1+\omega^2 t^2}\, dt$$

Subst.: $u = \operatorname{asinh}(\omega t)$

$t = \frac{1}{\omega} \sinh(u)$

$\frac{dt}{du} = \frac{1}{\omega} \cosh(u)$

$dt = \frac{1}{\omega} \cosh(u)\, du$

$$= c\omega \int_0^{\operatorname{asinh}(\omega T)} \frac{1}{\omega^2} \cosh^2(u)$$

$$= \frac{c}{\omega} \int_0^{\operatorname{asinh}(\omega T)} \cosh^2(u)$$

(siehe Blatt 4)

$$= \frac{1}{2} \cdot \frac{c}{\omega} \Big[\cosh(u) \sinh(u) + u\Big]_0^{\operatorname{asinh}(\omega T)}$$

$$= \frac{c}{2\omega} \Big[\sqrt{1+\omega^2 T^2} \cdot \omega T + \operatorname{asinh}(\omega T)\Big]$$

$$= \frac{cT}{2} \sqrt{1+\omega^2 T^2} + \frac{c}{2\omega} \operatorname{asinh}(\omega T)$$

Aufgabe 12

a) $v_x(t)$

$$\dot{v}_x(t) = -\alpha v_x$$

$$\frac{\dot{v}_x(t)}{v_x} = -\alpha$$

$$\frac{\frac{dv}{dt}}{v_x} = -\alpha$$

$$\frac{1}{v_x} dv = -\alpha\, dt \qquad \Big| \int$$

$$\ln|v_x| = -\alpha t$$

$$v_x(t) = v_{x0}\, e^{-\alpha t}$$

b) $v_y(t)$

$$\dot{v}_y(t) = -\alpha v_y - g \qquad \Big| + \alpha v_y$$

$$\dot{v}_y + \alpha v_y = -g$$

$$(\dot{v}_y + \alpha v_y)\, e^{\alpha t} = -g e^{\alpha t}$$

$$v_y = e^{-\alpha t} \cdot \int dt\, -g e^{\alpha t}$$

$$v_y(t) = e^{-\alpha t} \cdot \left(-\frac{g}{\alpha} e^{\alpha t} + C\right)$$

$$v_y(0) = 0 \iff C = \frac{g}{\alpha}$$

$$v_y(t) = \frac{g}{\alpha} e^{-\alpha t} - \frac{g}{\alpha}$$

c) zurückgelegter Weg

$$s = \int_0^T \sqrt{|\vec{v}|}\, dt \quad =: \int ds$$

$$= \int_0^T \sqrt{(v_{x0}\, e^{-\alpha t})^2 + \left(\frac{g}{\alpha} e^{-\alpha t} - \frac{g}{\alpha}\right)^2}\, dt$$

$$= \int_0^T \sqrt{v_{x0}^2 \cdot (e^{\alpha t})^{-2} + \left(\frac{g}{\alpha} e^{-\alpha t} - \frac{g}{\alpha}\right)^2}\, dt$$

$$= \int_0^T \sqrt{\left(\frac{v_{x0}}{e^{\alpha t}}\right)^2 + \left(\frac{g}{\alpha} e^{-\alpha t} - \frac{g}{\alpha}\right)^2}\, dt$$

$$= \int_0^T \sqrt{\left(\frac{v_{x0}}{e^{\alpha t}}\right)^2 + \left(\frac{e^{-\alpha t} - 1}{\frac{\alpha}{g}}\right)^2}\, dt$$

1. Substitution: $z = e^{-\alpha t}$; $t = -\frac{1}{\alpha} \ln|z|$;

$$\frac{dz}{dt} = -\alpha e^{-\alpha t}$$

$$dt = \frac{1}{-\alpha z}\, dz$$

$$\int_1^{e^{-\alpha T}} \sqrt{(v_{x0}\, z)^2 + \left(\frac{g(z-1)}{\alpha}\right)^2}\, dt$$

$$= \int_1^{e^{-\alpha T}} \sqrt{\frac{1}{\alpha^2 z^2}\left(v_{x0}^2 z^2 + \left(\frac{g(z-1)}{\alpha}\right)^2\right)}\, dz$$

$$= \int_1^{e^{-\alpha t}} \sqrt{\frac{v_{x0}^2}{\alpha^2} + \frac{g^2 (z-1)^2}{\alpha^4 z^2}}\, dz$$

$$= \int_1^{e^{-\alpha t}} \sqrt{\frac{v_{x0}^2}{\alpha^2} + \left(\frac{g(z-1)}{\alpha^2 z}\right)^2}\, dz$$

$$= \int_1^{e^{-\alpha t}} \sqrt{\frac{g^2}{\alpha^4}\left(\frac{v_{x0}^2 \alpha^2}{g^2} + \frac{z-1}{z}\right)}\, dz$$

$$= \int_1^{e^{-\alpha t}} \frac{g}{\alpha^2} \sqrt{\frac{v_{x0}^2 \alpha^2}{g^2} + \frac{z-1}{z}}\, dz$$

$$\int_1^{e^{-\alpha T}} \frac{g}{\alpha^2} \sqrt{\frac{v_{x0}^2 \alpha^2}{g^2} + \frac{z-1}{z}} \, dz \quad ; \qquad \frac{v_{x0} \, \alpha}{g} = b$$

$$= \frac{g}{\alpha^2} \int_1^{e^{-\alpha T}} \sqrt{b^2 + \left(\frac{z-1}{z}\right)^2} \, dz$$

2. Substitution: $z = \dfrac{1}{1 - b\omega}$

$$\omega = \frac{1}{b}\left(1 - \frac{1}{z}\right) = \frac{1}{b} - \frac{1}{zb}$$

$$\frac{dz}{d\omega} = -\frac{b}{(1-b\omega)^2}$$

$$dz = -\frac{b}{(1-b\omega)^2} \, d\omega$$

$$- \frac{g}{\alpha^2} \int_0^{\frac{1}{b}(1-e^{\alpha T})} \sqrt{b^2 + (b\omega)^2} \; \frac{b}{(1-b\omega)^2} \, d\omega$$

$$= - \frac{g}{\alpha^2} b^2 \int_0^{\frac{1}{b}(1-e^{\alpha T})} \sqrt{1 + \omega^2} \cdot \frac{1}{(1-b\omega)^2} \, d\omega$$

$$= \frac{g}{\alpha^2} b^2 \int_0^{\frac{1}{b}(1-e^{\alpha T})} \sqrt{1 + \omega^2} \cdot \frac{1}{(1-b\omega)^2} \, d\omega$$

partielle Integration: $u = \sqrt{1+\omega^2}$ $\quad v = \dfrac{1}{b(1-b\omega)}$

$u' = \dfrac{\omega}{\sqrt{1+\omega^2}}$ $\quad v' = \dfrac{1}{(1-b\omega)^2}$

$$= - \frac{g}{\alpha^2} b \left[\left[\frac{\sqrt{1+\omega^2}}{1 - b\omega} \right]_0^{\frac{1}{b}(1-e^{\alpha T})} - \underbrace{\int_0^{\frac{1}{b}(1-e^{\alpha T})} \frac{\omega}{\sqrt{1+\omega^2}\,(1-b\omega)} \, d\omega}_{I} \right]$$

Lösen von I:

$$\int_0^{\frac{1}{6}(1-e^{-\alpha T})} \frac{\omega}{\sqrt{1+\omega^2}\,(1-6\omega)}\, d\omega$$

3. Substitution: $\omega = \sinh(u)$

$$u = \operatorname{arsinh}(\omega)$$

$$\frac{du}{d\omega} = \frac{1}{\sqrt{1+\omega^2}}$$

$$d\omega = \sqrt{1+\omega^2}\, du$$

$$\int_0^{\operatorname{arsinh}\left(\frac{1}{6}(1-e^{\alpha T})\right)} \frac{\sinh u}{1-6\sinh u}\, du$$

4. Substitution: $v = \tanh\left(\frac{u}{2}\right) = \frac{\sinh u}{\cosh u + 1}$

$$u = 2\operatorname{artanh}(v)$$

$$\frac{du}{dv} = \frac{2}{1-v^2} \qquad \frac{dv}{du} = 1 - \tanh\left(\frac{u}{2}\right)$$

$$\sinh(2u) = 2\sinh u \cosh u$$

$$= 2\sin(\operatorname{artanh} v)\cosh(\operatorname{artanh} v)$$

$$= 2\frac{v}{\sqrt{1-v^2}}\frac{1}{\sqrt{1-v^2}} = \frac{2v}{1-v^2}$$

Grenze: $\tanh\left(\frac{1}{2}\operatorname{arsinh}\left(\frac{1}{6}(1-e^{\alpha T})\right)\right) = G$

$$\int_0^{G} \frac{2v}{(1-v^2)\left(1-6\frac{2v}{1-v^2}\right)} \cdot \frac{2}{1-v^2}\, dv$$

$$= \int_0^G \frac{4v}{(1-v)(1+v)(-v^2-2bv+1)}\, dv$$

Nullstellen: $$v_{1,2} = \frac{2b \pm \sqrt{4b^2+4}}{-2} = \frac{2b \pm \sqrt{4(1+b^2)}}{-2}$$

$$v_1 = -b - \sqrt{1+b^2} \qquad v_2 = -b + \sqrt{1+b^2}$$

$$\int_0^G \frac{4v}{-(1-v)(1+v)\left(v+b+\sqrt{1+b^2}\right)\left(v+b-\sqrt{1+b^2}\right)}\, dv$$

Partialbruchzerlegung:

$$\frac{4v}{(v-1)(1+v)\left(v+b+\sqrt{1+b^2}\right)\left(v+b-\sqrt{1+b^2}\right)}$$

$$= \frac{A}{v-1} + \frac{B}{1+v} + \frac{C}{v+b+\sqrt{1+b^2}} + \frac{D}{v+b-\sqrt{1+b^2}}$$

$$A = \lim_{v\to 1} \left(\frac{4v(v-1)}{(v-1)(1+v)\left(v+b+\sqrt{1+b^2}\right)\left(v+b-\sqrt{1+b^2}\right)} \right)$$

$$= \frac{-4}{2\left(1+b+\sqrt{1+b^2}\right)\left(1+b-\sqrt{1+b^2}\right)}$$

$$= \frac{-2}{\left(1+b+\sqrt{1+b^2}\right)\left(1+b-\sqrt{1+b^2}\right)}$$

$$= -\frac{1}{b}$$

$$B = \lim_{v\to -1} \left(\frac{4v}{(v-1)\left(v+b+\sqrt{1+b^2}\right)\left(v+b-\sqrt{1+b^2}\right)} \right)$$

$$= \frac{-2}{\left(-1+b+\sqrt{1+b^2}\right)\left(-1+b-\sqrt{1+b^2}\right)}$$

$$= -\frac{1}{b}$$

$$C = \lim_{v \to -6-\sqrt{1+6^2}} \left(\frac{-4v}{(1-v)(1+v)(v+6-\sqrt{1+6^2})} \right)$$

$$= \frac{-4(6+\sqrt{1+6^2})}{(1+6+\sqrt{1+6^2})(1-6-\sqrt{1+6^2})(-2\sqrt{1+6^2})}$$

$$= \frac{1}{6\sqrt{1+6^2}}$$

$$D = \lim_{v \to -6+\sqrt{1+6^2}} \left(\frac{-4v}{(1+v)(1-v)(v+6+\sqrt{1+6^2})} \right)$$

$$= \frac{-2(-6+\sqrt{1+6^2})}{-2(6\sqrt{1+6^2}-6^2)\sqrt{1+6^2}}$$

$$= -\frac{1}{6\sqrt{1+6^2}}$$

Integral:

$$\int_0^G \frac{4v}{(v-1)(1+v)(-v^2-26v+1)} \, dv$$

$$= \int_0^G -\frac{1}{6(v-1)} + \frac{1}{6(1+v)} + \frac{1}{6\sqrt{1+6^2}(v+6+\sqrt{1+6^2})} - \frac{1}{6\sqrt{1+6^2}(v+6-\sqrt{1+6^2})} \, dv$$

$$= \left[\frac{1}{6} \ln \left| \frac{v-1}{1+v} \right| - \frac{1}{6\sqrt{1+6^2}} \ln \left| \frac{v+6-\sqrt{1+6^2}}{v+6+\sqrt{1+6^2}} \right| \right]_0^G$$

$$s = -\frac{g}{\alpha^2} b \left[\left[\frac{\sqrt{1+\omega^2}}{1-b\omega} \right]_0^{\frac{1}{b}(1-e^{\alpha T})} - \left[\frac{1}{b} \ln\left| \frac{1-v}{1+v} \right| - \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{v+b-\sqrt{1+b^2}}{v+b+\sqrt{1+b^2}} \right| \right]_0^G \right]$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 - \frac{1}{b} \ln\left| \frac{1-G}{1+G} \right| + \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{G+b-\sqrt{1+b^2}}{G+b+\sqrt{1+b^2}} \right| - \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{b-\sqrt{1+b^2}}{b+\sqrt{1+b^2}} \right| \right]$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 - \frac{1}{b} \ln\left| \frac{1-G}{1+G} \right| + \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{G+b-\sqrt{1+b^2}}{G+b+\sqrt{1+b^2}} \right| - \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{b-\sqrt{1+b^2}}{b+\sqrt{1+b^2}} \right| \right]$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 + \frac{1}{b} \ln\left| \frac{1+G}{1-G} \right| - \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{G+b+\sqrt{1+b^2}}{G+b-\sqrt{1+b^2}} \right| + \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{b+\sqrt{1+b^2}}{b-\sqrt{1+b^2}} \right| \right]$$

$$\operatorname{artanh}(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x} \right)$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 + \frac{2}{b} \operatorname{artanh}(G) - \frac{1}{b\sqrt{1+b^2}} \left(\ln\left| \frac{1+\frac{\sqrt{1+b^2}}{G+b}}{1-\frac{\sqrt{1+b^2}}{G+b}} \right| - \ln\left| \frac{1+\frac{\sqrt{1+b^2}}{b}}{1-\frac{\sqrt{1+b^2}}{b}} \right| \right) \right]$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 + \frac{2}{b} \operatorname{artanh}(G) - \frac{1}{b\sqrt{1+b^2}} \left(\ln\left| \frac{G(b-\sqrt{1+b^2})-1}{G(b+\sqrt{1+b^2})-1} \right| \right. \right.$$

$$s = -\frac{g}{\alpha^2} b \left[\frac{\sqrt{1+\frac{1}{b^2}(1-e^{\alpha T})^2}}{e^{\alpha T}} - 1 + \frac{1}{b} \operatorname{asinh}\left(\frac{1}{b}(1-e^{\alpha T}) \right) - \frac{1}{b\sqrt{1+b^2}} \ln\left| \frac{\tanh\left(\frac{1}{2}\operatorname{asinh}\left(\frac{1}{b}(1-e^{\alpha T})\right)\right)(b-\sqrt{1+b^2})-1}{\tanh\left(\frac{1}{2}\operatorname{asinh}\left(\frac{1}{b}(1-e^{\alpha T})\right)\right)(b+\sqrt{1+b^2})-1} \right| \right.$$