Institute of Theoretical Particle Physics

Classical Theoretical Physics I

WS 2023

Exercise Sheet 3
Start: 10.11.2023
Due: 17.11.2023

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Please put your name on each sheet of your solution. Please put your tutoring session information on the front page (location, time, name of the tutor, etc.)

Problem 5: At time $t = 0$, an ant is at the location $x_0 \geq 0$ of a rubber band clamped at $x = 0$. The length of the band is $L(t) = L_0 + v_G t$, i.e. it is stretched at the constant speed v_G. The ant runs towards the end of the rubber band at v_A. Parameters included in the competence of other KIT faculties (lifespan of the ant, length of the rubber band) are set to ∞.

a) (1 point) Verify that in the time interval $[t, t + dt]$, the displacement traveled by the ant is $dx = v_A dt + v_G \dfrac{x(t)}{L(t)} dt$. Consider $r(t) = x(t)/L(t)$ and express $\dot{r}$ in terms of L_0, v_G, and v_A.

b) (1 point) Calculate $r(t)$. (Pay attention to the initial condition $r_0 = x_0/L_0$.) Specify at what time $t = T$ the ant reaches the endpoint $x = L$

c) (1 point) Consider the case $L_0 = 1\,m$, $v_G = 1\,m/s$, $v_A = 1\,cm/s$.
When does the ant, starting from $x_0 = 0$, reach the end? Provide T in multiples of the age of the universe, which is 13.8×10^9 years.
To what extent does the situation improve if the ant starts from $x_0 = L_0/2 = 0.5\,m$?

d) (1 point) For $x_0 = 0$, we now consider a discretized version of the problem: In the interval $[(n-1)\Delta t, n\Delta t]$, $n \in \mathbb{N}$, the ant first moves by a distance of Δx, and then (at time $n\Delta t$) the rubber band is instantly stretched from length nL_0 to $(n+1)L_0$. Let x_n denote the position of the ant after the n-th step (but before the subsequent stretching of the band), and we consider $r_n = x_n/(nL_0)$, i.e., after the first step, $r_1 = \Delta x/L_0$. What progress $\Delta r_k = r_k - r_{k-1}$ (for $k \geq 2$) does the ant make in the k-th step? Show that r_n is proportional to the *harmonic sum*: $r_n = \dfrac{\Delta x}{L_0} \displaystyle\sum_{k=1}^{n} \dfrac{1}{k}$.

e) (1 point) While integrals come easily to physicists, sums often pose difficulties for them. The *Euler-Maclaurin formula* allows one to approximate a sum through an integral:

$$\sum_{k=1}^{n} f(k) \;=\; \int_1^n dx f(x) \;+\; \frac{f(n) + f(1)}{2} \;+\; \sum_{k=1}^{l} \frac{B_{2k}}{(2k)!} \left(f^{(2k-1)}(n) - f^{(2k-1)}(1) \right) + R_{2l}, \qquad (1)$$

where the remainder term R_{2l} is neglected in the approximation. $f^{(j)}$ represents the j-th derivative of f, and the *Bernoulli numbers* B_j are $B_2 = 1/6$, $B_4 = -1/30$, and so on. We consider $r_3 = \dfrac{11}{6} \dfrac{\Delta x}{L_0}$ and $r_{20} = 3.59774 \dfrac{\Delta x}{L_0}$. Compute the approximations from Gl. (1) for both values, considering the cases $l = 0$, $l = 1$, and $l = 2$.
Note: You can solve parts (d) and (e) even if you haven't worked on (a)-(c).

Problem 6: We consider a vehicle with mass m that, at time $t = 0$, is moving with velocity v_0 and is decelerated by air friction for $t \geq 0$. The velocity $v(t)$ satisfies the differential equation

$$\dot{v} = -\alpha v - \beta v^2, \qquad \alpha, \beta \geq 0. \qquad (2)$$

$m\alpha v$ and $m\beta v^2$ are the magnitudes of the *Stokes's* and the *Newton's* friction forces.

a) (1 Point) Compute the partial fraction decomposition of $\dfrac{1}{\alpha v + \beta v^2}$ for $\alpha \neq 0$.

b) (2 Points) Determine $v(t)$. Take care when integrating to ensure that the argument of the logarithm is dimensionless. Express the integration constant in terms of v_0.
Draw $v(t)$ for $0 \leq t \leq 20$ s for the case $v_0 = 36$ m/s, $\alpha = 2.5 \cdot 10^{-2}$ s^{-1}, $\beta = 4 \cdot 10^{-3}$ m^{-1}. Include in the same plot the solutions for the cases when $\alpha = 0$ and $\beta = 0$.
Note: You can proceed similarly to Eq. (10) in the lecture. Consider the cases $\alpha = 0$ and $\alpha \neq 0$ separately.

c) (2 Points) Determine $x(t)$ with the initial condition $x(0) = x_0$.
Draw the distance $x(\infty) - x_0$ that the vehicle needs to roll out as a function of α for $0 < \alpha \leq 2.5 \cdot 10^{-2}$ s^{-1} for the values of v_0 and β given in b).
Note: Substitute $z = e^{-\alpha t}$ to solve the integral over $v(t)$.