Institute of Theoretical Particle Physics

Classical Theoretical Physics I

WS 2023

Exercise Sheet 5
Start: 24.11.2023
Due: 01.12.2023

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Please put your name on each sheet of your solution. Please put your tutoring session information on the front page (location, time, name of the tutor, etc.)

Problem 9: A particle of mass m is subjected to Stokesian frictional force and an external periodic force $mb\sin(\Omega t)$. The velocity $v(t)$ satisfies

$$\dot{v} = -\alpha v + b\sin(\Omega t), \qquad \alpha, b, \Omega > 0. \tag{1}$$

a) (1 Point) Determine the antiderivative (including integration constant) of $\sin(\Omega t)e^{\alpha t}$.
Hint: Use $\sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$ and express the result in terms of $\sin(\Omega t)$ and $\cos(\Omega t)$.

b) (2 Points) Solve (1) with the initial condition $v(0) = 0$. Consider only the case $v(t) \geq 0$.
Hint: You can directly use a formula from the lecture.

c) (1 Point) Determine the distance traveled $x(t)$ for $x(0) = 0$.

d) (1 Point) Gl. (1) only makes sense for $v \geq 0$. Verify that $v(t)$ is initially positive for $t > 0$, and determine the time T with $v(T) = 0$ in the case $\alpha \gg \Omega$. Determine $x(T)$.

Problem 10: At time $t = 0$, an ant is at the location $x_0 \geq 0$ of a rubber band clamped at $x = 0$. The length of the band is $L(t) = L_0 + v_G t$, i.e. it is stretched at the constant speed v_G. The ant runs towards the end of the rubber band at v_A. Parameters included in the competence of other KIT faculties (lifespan of the ant, length of the rubber band) are set to ∞.

a) (1 point) Verify that in the time interval $[t, t + dt]$, the displacement traveled by the ant is $dx = v_A dt + v_G \frac{x(t)}{L(t)} dt$. Consider $r(t) = x(t)/L(t)$ and express $\dot{r}$ in terms of L_0, v_G, and v_A.

b) (1 point) Calculate $r(t)$. (Pay attention to the initial condition $r_0 = x_0/L_0$.) Specify at what time $t = T$ the ant reaches the endpoint $x = L$

c) (1 point) Consider the case $L_0 = 1\,\text{m}$, $v_G = 1\,\text{m/s}$, $v_A = 1\,\text{cm/s}$.
When does the ant, starting from $x_0 = 0$, reach the end? Provide T in multiples of the age of the universe, which is 13.8×10^9 years.
To what extent does the situation improve if the ant starts from $x_0 = L_0/2 = 0.5\,\text{m}$?

d) (1 point) For $x_0 = 0$, we now consider a discretized version of the problem: In the interval $[(n-1)\Delta t, n\Delta t]$, $n \in \mathbb{N}$, the ant first moves by a distance of Δx, and then (at time $n\Delta t$) the rubber band is instantly stretched from length nL_0 to $(n+1)L_0$. Let x_n denote the position of the ant after the n-th step (but before the subsequent stretching of the band), and we consider $r_n = x_n/(nL_0)$, i.e., after the first step, $r_1 = \Delta x/L_0$. What progress $\Delta r_k = r_k - r_{k-1}$ (for $k \geq 2$) does the ant make in the k-th step? Show that r_n is proportional to the *harmonic sum*: $r_n = \frac{\Delta x}{L_0} \sum_{k=1}^{n} \frac{1}{k}$.

e) (1 point) While integrals come easily to physicists, sums often pose difficulties for them.

The *Euler-Maclaurin formula* allows one to approximate a sum through an integral:

$$\sum_{k=1}^{n} f(k) \;=\; \int_1^n dx f(x) + \frac{f(n)+f(1)}{2} + \sum_{k=1}^{l} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(1)\right) + R_{2l}, \quad (2)$$

where the remainder term R_{2l} is neglected in the approximation. $f^{(j)}$ represents the j-th derivative of f, and the *Bernoulli numbers* B_j are $B_2 = 1/6$, $B_4 = -1/30$, and so on. We consider $r_3 = \frac{11}{6}\frac{\Delta x}{L_0}$ and $r_{20} = 3.59774\frac{\Delta x}{L_0}$. Compute the approximations from Gl. (2) for both values, considering the cases $l = 0$, $l = 1$, and $l = 2$.
Note: You can solve parts (d) and (e) even if you haven't worked on (a)-(c).