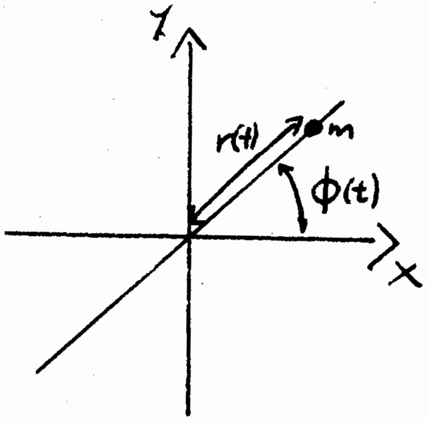
1)

$$x = r\cdot\cos(\phi)$$
$$y = r\cdot\sin(\phi)$$
$$\dot{x} = \dot{r}\cos(\phi) - r\dot{\phi}\sin(\phi)$$
$$\dot{y} = \dot{r}\sin(\phi) + r\dot{\phi}\cos(\phi)$$



$x = x(t) \quad y = y(t)$
$r = r(t) \quad \phi = \phi(t)$

a) $\mathcal{L} = T - U \;;\; T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \;;\; U \equiv 0$

$$\Rightarrow \mathcal{L}(r, \dot{r}, \phi, \dot{\phi}) = \frac{1}{2}m\left((\dot{r}\cos(\phi) - r\dot{\phi}\sin(\phi))^2 + (\dot{r}\sin(\phi) + r\dot{\phi}\cos(\phi))^2\right)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) \quad 1p$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) = \underline{\underline{\frac{d}{dt}(mr^2\dot{\phi}) = 0}} = \frac{\partial\mathcal{L}}{\partial\phi} \quad 0{,}5p \qquad (1)$$

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{d}{dt}(m\dot{r}) = \underline{\underline{m\ddot{r} = mr\dot{\phi}^2}} = \frac{\partial\mathcal{L}}{\partial r} \qquad (2) \quad 0{,}5p$$

Aus (1) folgt, dass $L = mr^2\dot{\phi}$, der Drehmoment erhalten ist. 0,5p

$$\Rightarrow \quad \dot{\phi} = \frac{L}{mr^2}$$

$$\overset{(2)}{\Rightarrow} \quad \cancel{m}\ddot{r} = \cancel{m}r\dot{\phi}^2 = mr\frac{L^2}{m^2r^4} \quad 0{,}5p$$

$$\Rightarrow \quad \ddot{r} = \frac{L^2}{m^2}\,\frac{1}{r^3} \quad \Rightarrow \quad \underline{\underline{\ddot{r}r^3 = \frac{L^2}{m^2} = \beta = \text{konst.}}}$$

b) $r(t) = \sqrt{a(t+b)^2 + c} \quad \Rightarrow \quad r^3 = (a(t+b)^2 + c)^{3/2}$

$$\Rightarrow \dot{r}(t) = \frac{\cancel{2}a(t+b)}{\cancel{2}\sqrt{a(t+b)^2 + c}} = a(t+b)\cdot(a(t+b)^2 + c)^{-\frac{1}{2}}$$

$$\Rightarrow \ddot{r}(t) = a(a(t+b)^2 + c)^{-\frac{1}{2}} - a^2(t+b)^2(a(t+b)^2 + c)^{-\frac{3}{2}}$$

$$\Rightarrow \ddot{r} r^3 = a(a(t+b)^2+c) - a^2(t+b)^2 = a\cdot c = B$$

$$\Rightarrow \quad c = \frac{B}{a}$$

1,5 P

$$\Rightarrow r(t) = \sqrt{a(t+b)^2 + \frac{B}{a}} \quad ; \quad B = \frac{L^2}{m^2}$$

$$L = m r^2 \dot{\phi}$$

$$\Rightarrow \quad \dot{\phi} = \frac{L}{m r^2} = \sqrt{B}\, \frac{1}{a(t+b)^2 + \frac{B}{a}}$$

$$\Rightarrow \phi(t) - \phi_0 = \int \dot{\phi}(t)\,dt = \frac{\sqrt{B}}{a} \int \frac{1}{(t+b)^2 + \frac{B}{a^2}}$$

$$x = t+b \Rightarrow \frac{dx}{dt} = 1$$

$$\Rightarrow \phi(t) - \phi_0 = \frac{\sqrt{B}}{a} \int \frac{dx}{x^2 + \frac{\sqrt{B}^2}{a^2}} = \frac{\sqrt{B}}{a} \frac{a}{\sqrt{B}} \arctan\left(\frac{(t+b)}{\frac{\sqrt{B}}{a}}\right)$$

1,5 P

$$\Rightarrow \phi(t) = \arctan \frac{a(t+b)}{\sqrt{B}} + \phi_0$$

c) $\dot{\phi}(0) = \frac{\sqrt{B}}{ab^2 + \frac{B}{a}} = \omega_0$ z.z.: $\omega_0 t = \frac{a}{\sqrt{B}} t + \frac{b}{\sqrt{B}} \quad \forall t$

$$\Rightarrow b = 0 \; ; \; \omega_0 = \frac{a}{\sqrt{B}}$$

$$\phi(0) = \underbrace{\arctan \underbrace{\frac{ab}{\sqrt{B}}}_{=0}}_{=0} + \phi_0 = 0 \Rightarrow \phi_0 = 0$$

$$\dot{r}(0) = ab\left(ab^2 + \frac{B}{a}\right)^{-\frac{1}{2}} = 0 \quad \checkmark$$

$$r(0) = \sqrt{ab^2 + \frac{B}{a}} = r_0 = \sqrt{\frac{B}{a}}$$

Fortsetzung 1c) $\frac{B}{r_0^4} = \frac{a}{r_0^2} = \frac{a^2}{B} = \omega$

$$\rightarrow \quad \phi(t) = \arctan(\omega_0 t)$$ (2P)

$$r(t) = \sqrt{a t^2 + \frac{B}{a}} = \sqrt{\frac{B}{r_0^2} t^2 + r_0^2}$$

$$= r_0 \sqrt{1 + \frac{B}{r_0^4} t^2} = r_0 \sqrt{1 + \omega_0^2 t^2}$$

$$\tan(\phi) = \omega_0 t = \frac{y}{x} \Rightarrow y = \omega_0 t x$$

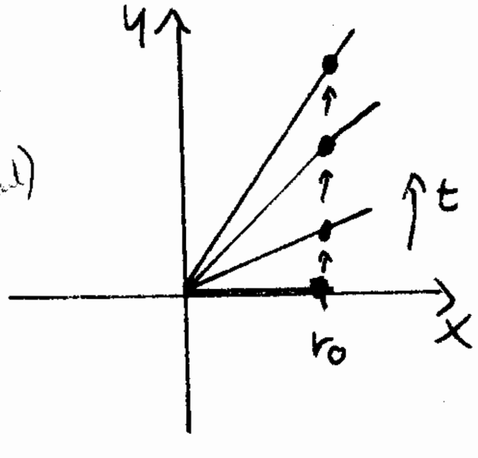
$$r^2 = x^2 + y^2 \Rightarrow r_0^2 (1 + \omega_0^2 t^2) = x^2 + y^2$$

$$\Rightarrow x^2 + \omega_0^2 t^2 x^2 = x^2 (1 + \omega_0 t^2) = r_0^2 (1 + \omega_0 t^2)$$

$$\Rightarrow x = (\pm) r_0 \Rightarrow y = \omega_0 t x = r_0 \omega_0 t$$ (0,5P)

Die Masse startet aufgrund der Anfangsbed. ($\phi_0 = 0$) auf der x-Achse, bei $x(t) = x_0 = r_0$. Die Stange dreht sich dann gegen den Uhrzeigersinn, wobei die Masse wegen der auf sie wirkenden Zentrifugalkraft vom Urspung weg driftet. Die y-koordinate wächst also an, während die x-koordinate konstant bleibt.

Es handelt sich um ein freies Teilchen (konstante Geschwindigkeit)



2) $H(x,p_x,y,p_y) = \frac{p_x p_y}{2m} + 2m\omega^2 xy$

a)

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_y}{2m} \Rightarrow p_y = 2m\dot{x} \quad (1)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -2m\omega^2 y \Rightarrow y = -\frac{\dot{p}_x}{2m\omega^2} \quad (2)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_x}{2m} \Rightarrow p_x = 2m\dot{y} \quad (3)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -2m\omega^2 x \Rightarrow x = -\frac{\dot{p}_y}{2m\omega^2} \quad (4)$$

$$\Rightarrow \dot{p}_y = 2m\ddot{x} = -2m\omega^2 x \Rightarrow \ddot{x} + \omega^2 x = 0$$

$$\dot{p}_x = 2m\ddot{y} = -2m\omega^2 y \Rightarrow \ddot{y} + \omega^2 y = 0$$

Es handelt sich um gewöhnliche harmonische, ungedämpfte und ungetriebene Oszillatoren.

$$\Rightarrow x(t) = A_x \sin(\omega t) + B_x \cos(\omega t)$$

$$y(t) = A_y \sin(\omega t) + B_y \cos(\omega t)$$

3

b) $H = \sum_k p_k q_k - \mathcal{L} \Rightarrow \mathcal{L} = \sum_k p_k q_k - H$

$$\Rightarrow \mathcal{L} = p_x \frac{p_y}{2m} + p_y \frac{p_x}{2m} - \frac{p_x p_y}{2m} - 2m\omega^2 xy$$

$$= \frac{p_x p_y}{2m} - 2m\omega^2 xy$$

$$= \frac{4m^2 \dot{x}\dot{y}}{2m} - 2m\omega^2 xy$$

$$= \underline{\underline{2m\dot{x}\dot{y} - 2m\omega^2 xy = \mathcal{L}(x,\dot{x},y,\dot{y})}}$$

2

c) $\{A,B\} = \sum_i \left[\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i}\right]$

$$R = x^2 + \frac{p_y^2}{4m^2\omega^2} \neq R(t)$$

$$\Rightarrow \frac{dR}{dt} = \underbrace{\frac{\partial R}{\partial t}}_{=0} + \{H,R\} \checkmark$$

$$= \sum_{i=x,y}\left[\frac{\partial H}{\partial p_i}\frac{\partial R}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial R}{\partial p_i}\right] \checkmark$$

$$= \frac{p_y}{\cancel{2}m}\cdot\cancel{2}x - \cancel{2m\omega^2 y\cdot 0} + \cancel{\frac{p_x}{2m}\cdot 0} - \cancel{2}m\omega^2 x\frac{\cancel{2}p_y}{\cancel{4}m^2\omega^2}$$

$$= \frac{p_y x}{m} - \frac{p_y x}{m} = 0 \checkmark$$

2

$\Rightarrow$ R ist erhalten (da $\frac{dR}{dt} = \dot{R} = 0$) ✓

7,0

3)

a) Die Eigenwerte λ_i einer (reellen) symmetrischen Matrix sind reell; die zugehörigen Eigenvektoren $\vec{a}^{(i)}$ sind paarweise orthogonal. ✓ 1

b)

$$M = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

<u>Eigenwerte:</u>

$\det(M - \lambda E) = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 & 1 \\ -2 & 4-\lambda & 2 \\ 1 & 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2(4-\lambda) - 4 - 4 - 4(3-\lambda) - 4(3-\lambda) - (4-\lambda) = 0$$

$$\Rightarrow (3-\lambda)^2(4-\lambda) - 8 - 8(3-\lambda) - (4-\lambda) = 0$$

z.z.: $\lambda_1 = 0$; $\lambda_2 = 4$; $\lambda_3 = 6$ Eigenwerte

<u>$\lambda_1 = 0$ einsetzen:</u>

$3^2 \cdot 4 - 8 - 8 \cdot 3 - 4 = 36 - 8 - 24 - 4 = 0$ ✓

<u>$\lambda_2 = 4$ einsetzen:</u>

$(-1)^2 \cdot 0 - 8 - 8(-1) - 0 = -8 + 8 = 0$ ✓

<u>$\lambda_3 = 6$ einsetzen:</u>

$(-3)^2 \cdot (-2) - 8 - 8(-3) - (-2) = -18 - 8 + 24 + 2 = 0$ ✓ qed.

Da es sich um eine (3x3)-Matrix handelt, sind dies <u>alle</u> Eigenwerte. ✓ 2

Eigenvektoren:

$\lambda_1 = 0$

$\Rightarrow M \cdot \vec{a} = 0$

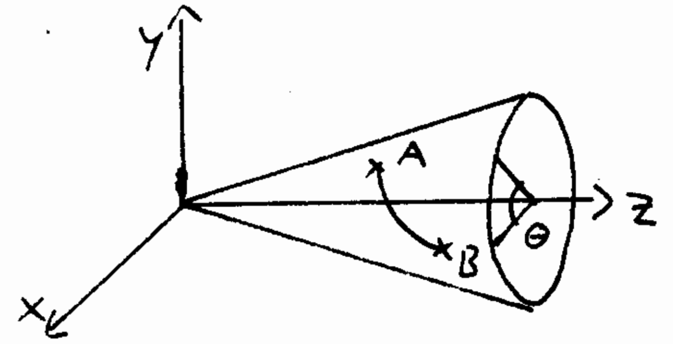
$$\Rightarrow \left(\begin{array}{ccc|c} 3 & -2 & 1 & 0 \\ -2 & 4 & 2 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow{+2,\ -3} \left(\begin{array}{ccc|c} 0 & -8 & -8 & 0 \\ 0 & 8 & 8 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow{+,\ 1:8}$$

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$\Rightarrow \quad a_2 = -a_3 \quad ; \quad a_1 = -a_3$

$$\Rightarrow \quad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \underline{\underline{\vec{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}} \quad ✓ \quad 2$$

4)



a)

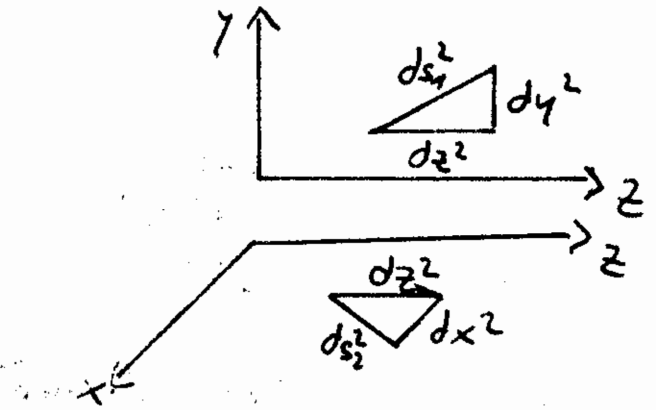
$$x = z \cdot \cos\theta$$
$$y = z \cdot \sin\theta$$

$$\Rightarrow \quad \frac{dx}{dz} = \cos\theta \; ; \quad \frac{dx}{d\theta} = -z \cdot \sin\theta \;\; \longrightarrow dx = \underbrace{\cos\theta dz - z\sin\theta d\theta}_{dx^*}$$

$$\frac{dy}{dz} = \sin\theta \; ; \quad \frac{dy}{d\theta} = z \cdot \cos\theta \;\; \longrightarrow dy = \underbrace{\sin\theta dz + z\cos\theta d\theta}_{dy^*}$$

½

Betrachte zunächst nur xz- bzw. yz-Ebene:



$$(ds_1)^2 = (dy^*)^2 + (dz)^2$$

$$(ds_2)^2 = (dx^*)^2 + (dz)^2$$

$$(ds)^2 = (ds_1)^2 + (ds_2)^2 = 2(dz)^2 + (dx^*)^2 + (dy^*)^2$$
$$= 2(dz)^2 + (z^2 \cdot \sin^2\theta) \cdot (d\theta)^2 + (z^2 \cdot \cos^2\theta) (d\theta)$$
$$= \underline{\underline{2(dz)^2 + z^2 (d\theta)^2}}$$

½

$\rightarrow (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \ldots$

oder dx^*, dy^* definieren

1,5 P

b) $$S=\int_A^B ds = \int \sqrt{2(dz)^2+z^2(d\theta)^2} = \int \underbrace{\sqrt{2+z^2\cdot\theta'^2}}_{=f(z,\theta,\theta')}\,dz$$ ✓

Streckeminimieren $\Rightarrow \frac{d}{dz}\left(\frac{\partial f}{\partial\theta'}\right)=\frac{\partial f}{\partial\theta}$ $\quad$ 1

$$\Rightarrow \frac{d}{dz}\left(\frac{\cancel{2}\theta' z^2}{\cancel{2}\sqrt{2+z^2\cdot\theta'^2}}\right)=0$$

$$\Rightarrow \frac{\theta' z^2}{\sqrt{2+z^2\theta'^2}} = c = \text{konst.}$$ ✓

$$\Rightarrow \theta' z^2 = c\sqrt{2+z^2\theta'^2} \quad |^2$$ ✓

$$\Rightarrow \theta'^2(z^4-c^2z^2) = 2c^2$$ 1P

$$\Rightarrow \theta' = \frac{\sqrt{2}c}{\sqrt{z^4-c^2z^2}} \overset{\cancel{c=1}}{=} c\frac{\sqrt{2}}{\sqrt{z^4-z^2}}$$

Sub:
$z=cu$
$\frac{dz}{du}=c$

$$\Rightarrow \theta(z)-\theta_0 = \int\theta'(z)\,dz = \sqrt{2}\int\frac{1}{\sqrt{z^4-z^2}}dz$$

$\theta(t)-\theta_0=\sqrt{2}\arctan\sqrt{z^2-1}$

$$=\sqrt{2}\arctan\sqrt{z^2-1}$$ ✓ $\quad$ 0,5P

$$\Rightarrow \underline{\underline{\theta(z)=\sqrt{2}\arctan\sqrt{\tfrac{z^2}{c^2}-1}+\theta_0}}$$

$$\Rightarrow \arctan\sqrt{\tfrac{z^2}{c^2}-1} = \frac{1}{\sqrt{2}}(\theta-\theta_0)$$

$$\Rightarrow \sqrt{\tfrac{z^2}{c^2}-1} = \tan\left(\frac{1}{\sqrt{2}}(\theta-\theta_0)\right)$$

$$\Rightarrow z^2 = c^2\tan^2\Big(\underbrace{\tfrac{1}{\sqrt{2}}(\theta-\theta_0)}_{=\Omega}\Big)+1$$

$$\Rightarrow z(\theta) = c\sqrt{\tan^2(\Omega)+1}$$

$$z(\theta) = c\sqrt{\frac{\sin^2(\Omega)}{\cos^2(\Omega)} + 1}$$

$$= c\sqrt{\frac{\sin^2(\Omega)+\cos^2(\Omega)}{\cos^2(\Omega)}} = \frac{1}{\cos(\Omega)} \cdot c$$

$$= \underline{\underline{\frac{1}{\cos\left(\frac{1}{\sqrt{2}}(\theta - \theta_0)\right)} \cdot c}}$$

$\underline{1^P}$