Exercise 4: Hamiltonian mechanics — 22 points

A relativistic free particle of rest-mass m in a one-dimensional space is described by the Lagrangian

$$L = -mc^2\sqrt{1-\frac{v^2}{c^2}} \, , \tag{1}$$

where $v = \dot{x}$.

(a) [4 pt] Show that the Hamiltonian for the relativistic Lagrangian in Equation (1) is given by $H = c\sqrt{p^2+m^2c^2}$.

(b) [5 pt] Calculate the Poisson brackets $\{p, H\}$ and $\{x, H\}$. Use these results to find the explicit time dependence of $x(t)$.

The expansion of the Hamiltonian in question (a) around the non-relativistic limit allows for the computation of relativistic corrections. After the addition of a potential, it is similarly possible to study relativistic corrections to the harmonic oscillator. After an intricate canonical transformation, the resulting Hamiltonian is written as

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 + \lambda\left(\frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2\right)^2 \, . \tag{2}$$

(c) [3 pt] Derive the Hamilton equations of motion for the Hamiltonian in Equation (2).

(d) [3 pt] Show with the help of Poisson brackets that

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2 \tag{3}$$

is a conserved quantity.

(e) [3 pt] Using the result of the previous question, show that x satisfies

$$\ddot{x} + \omega^2 x = 0 \, , \tag{4}$$

where $\omega = \omega_0\left(1 + 2\lambda H_0\right)$.

(f) [4 pt] Solve the differential equation in Equation (4). Express ω in terms of λ, m, ω_0 and the amplitude of oscillation A.

Solution of exercise 1: Questions

(a) 1 pt *Independent of time*
The energy is conserved

(b) 1 pt *p_x conserved*
The Lagrangian is independent of x. Or:
The Lagrangian is invariant under translations in the x direction.

(c) 1 pt *Force with $\omega \approx \Omega$*
The amplitude will grow, as the system is close to resonance.

(d) 1 pt *What is described by $L = \frac{1}{2}m\dot{\vec{x}} + \frac{k}{r}$*
Kepler problem, Coulomb problem, Planet around sun, classical electron around nucleus, ...

(e) 2 pt *What is Liouville's theorem*
Areas in phase-space are conserved in time.

(f) 2 pt *Variation of action is zero*
The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial f}{\partial \dot{q}} = \frac{\partial f}{\partial q} \tag{5}$$

(g) 2 pt *Canonical transformation with $F = qQ$*
When F is a function of q and Q, we have

$$p = \frac{\partial F}{\partial q}\,, \qquad P = -\frac{\partial F}{\partial Q} \tag{6}$$

so the transformation is

$$Q = p\,, \qquad P = -q \tag{7}$$

(h) 3 pt *The square plate*

$$I = \sum mr^2 \tag{8}$$

so with respect to the two axes we get

$$\begin{aligned} I_z &= 4(m(\sqrt{2}a)^2) = 8ma^2 \\ I_{\text{diag}} &= 2(m(\sqrt{2}a)^2) + 2 \times 0 = 4ma^2 \end{aligned} \tag{9}$$

(i) 3 pt *Derive $t(x)$*

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + U(x) \Leftrightarrow \\ \frac{dx}{dt} &= \sqrt{\frac{2}{m}}\sqrt{E - U(x)} \Leftrightarrow \\ dt &= \sqrt{\frac{m}{2}}\frac{1}{\sqrt{E - U(x)}}dx \Leftrightarrow \\ t &= \sqrt{\frac{m}{2}}\int_{x_1}^{x_2}\frac{dx}{\sqrt{E - U(x)}} \end{aligned} \tag{10}$$

(j) 4 pt *Solve the oscillator with constant force*
The equation of motion is

$$m\ddot{x} + m\omega^2 x = F_0 \vartheta(t) \tag{11}$$

For $t > 0$ the general solution is

$$x = A\cos(\omega t + \theta) + \frac{F_0}{m\omega^2} \tag{12}$$

which corresponds to

$$\dot{x} = -\omega A \sin(\omega t + \theta) \tag{13}$$

The boundary conditions are $x(0) = \dot{x}(0) = 0$ implying

$$\theta = 0\ , \qquad A = -\frac{F_0}{m\omega^2} \tag{14}$$

and inserting gives

$$x = \frac{F_0}{m\omega^2}\left(1 - \cos(\omega t)\right) \tag{15}$$

Solution of exercise 2: ParticleCapture

(a) [4 pt] *What are the symmetries of the Lagrangian and what are the conserved quantities*
The symmetries are invariance under time-translations and under rotations. The corresponding conserved quantities are the energy and the (three components of the) angular momentum.

(b) [3 pt] *How to reduce to a 2d problem*
Since $\vec{M} = \vec{r} \times \vec{p}$, $\vec{r}$ will always be perpendicular to $\vec{M}$. We can pick the coordinate system such that the conserved $\vec{M}$ points along the z-axis, and then we get that $\vec{r}$ is confined to the two-dimensional xy plane.

(c) [3 pt] *Express E using U_{eff}*

$$\begin{aligned} E &= \tfrac{1}{2} m \dot{\vec{r}}^2 + U(r) \\ &= \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r) \\ &= \tfrac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + U(r) \\ &= \tfrac{1}{2} m \dot{r}^2 + U_{\text{eff}}(r) \qquad \text{with} \qquad U_{\text{eff}}(r) = \frac{M^2}{2mr^2} + U(r) \end{aligned} \tag{16}$$

along the way we used

$$\vec{M} = m r^2 \dot{\theta} \vec{u}_z \tag{17}$$

(d) [3 pt] *Re-express U_{eff} using ρ and E*
Having energy E at infinity, corresponds to the momentum $p_\infty = \sqrt{2mE}$. That gives the angular momentum $M = p_\infty \rho = \sqrt{2mE}\rho$. Inserting this in the expression for U_{eff} gives

$$\begin{aligned} U_{\text{eff}}(r) &= \frac{M^2}{2mr^2} + U(r) \\ &= \frac{E\rho^2}{r^2} + U(r) \end{aligned} \tag{18}$$

(e) [4 pt] *Sketch the potential and describe the types of motion*
We now have

$$U_{\text{eff}}(r) = \left(E\rho^2 + b\right) \frac{1}{r^2} - \frac{c}{r^4} \tag{19}$$

The sketch is shown in Figure 3. The types of motion are scattering, circular orbit, capture (from infinity), capture (while bound).

(f) [4 pt] *Calculate the maximum of U_{eff}*
The maximum is where the derivative is zero.

$$\frac{dU_{\text{eff}}}{dr} = \left(E\rho^2 + b\right) \frac{-2}{r^3} - \frac{-4c}{r^5} \tag{20}$$

and that is zero when

$$-4c = -2\left(E\rho^2 + b\right) r^2 \Leftrightarrow$$
$$r = \pm\sqrt{\frac{2c}{E\rho^2 + b}} \tag{21}$$

Inserting this point gives

$$U_{\text{eff}}\big|_{\max} = (E\rho^2 + b)\frac{E\rho^2 + b}{2c} - c\left(\frac{E\rho^2 + b}{2c}\right)^2$$
$$= \frac{(E\rho^2 + b)^2}{4c} \tag{22}$$

(g) 7 pt *Calculate the cross section for capture*
We derive an upper bound on ρ:

$$E > U_{\text{eff}}\big|_{\max} \Leftrightarrow E > \frac{(E\rho^2 + b)^2}{4c} \Leftrightarrow \sqrt{4cE} > E\rho^2 + b \Leftrightarrow$$
$$\rho^2 < \frac{2\sqrt{cE} - b}{E} \equiv \rho^2_{\max} \tag{23}$$

Since ρ is by definition positive, we get the refined condition on the minimum energy in terms of the parameters in the potential:

$$0 < \frac{2\sqrt{cE} - b}{E} \Leftrightarrow E > \frac{b^2}{4c} \tag{24}$$

The lower bound $\rho_{\min}$ is zero. The capture cross section is then $\pi\rho^2_{\max}$ if the energy is large enough, and zero otherwise:

$$\sigma_{\text{capture}} = \begin{cases} \pi\frac{2\sqrt{cE}-b}{E} & \text{for } E > \frac{b^2}{4c} \\ 0 & \text{for } E < \frac{b^2}{4c} \end{cases} \tag{25}$$

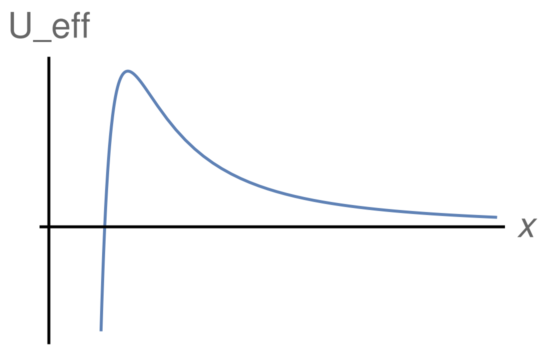

Figure 3: Sketch of U_{eff}.

Solution of exercise 3: SpringPendulumSystem

(a) $\boxed{4\text{ pt}}$ *Give a Lagrangian*

$$L = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\Big(\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\cos(\theta)\dot{x}\dot{\theta}\Big) - \tfrac{1}{2}kx^2 - mg\ell\Big(1-\cos(\theta)\Big)\ . \tag{26}$$

(b) $\boxed{4\text{ pt}}$ *Construct Euler-Lagrange equations*
Derive with respect to x and $\dot{x}$:

$$(M+m)\ddot{x} + m\ell\cos(\theta)\ddot{\theta} + kx - m\ell\sin(\theta)\dot{\theta}^2 = 0\ . \tag{27}$$

Derive with respect to θ and $\dot{\theta}$:

$$m\ell^2\ddot{\theta} + m\ell\cos(\theta)\ddot{x} + mg\ell\sin(\theta) = 0\ . \tag{28}$$

(c) $\boxed{3\text{ pt}}$ *Expand Lagrangian*
Set $M = m$ and $g = k\ell/(2m)$. Replace $(1-\cos(\theta)) = \theta^2/2$ and $\cos(\theta) = 1$ (because the latter is already multiplied by something small). The Lagrangian becomes

$$\begin{aligned} L &= \tfrac{1}{2}m\Big(2\dot{x}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{x}\dot{\theta}\Big) - \tfrac{1}{2}kx^2 - \tfrac{1}{2}k\ell^2\frac{\theta^2}{2} \\ &= \tfrac{1}{2}m\Big(2\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\Big) - \tfrac{1}{2}k\Big(q_1^2 + \tfrac{1}{2}q_2^2\Big) \\ &\equiv \tfrac{1}{2}\Big(m_{11}\dot{q}_1^2 + m_{22}\dot{q}_2^2 + (m_{12}+m_{21})\dot{q}_1\dot{q}_2\Big) - \tfrac{1}{2}\Big(k_{11}q_1^2 + k_{22}q_2^2\Big)\ . \end{aligned} \tag{29}$$

From this we read of that

$$\hat{m} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\ , \quad \hat{k} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = \frac{k}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\ . \tag{30}$$

(d) $\boxed{4\text{ pt}}$ *Derive eigenfrequencies*
Method 1: Set $\det(\hat{k} - \omega^2\hat{m}) = 0$. This gives quadratic equation for ω^2,

$$m^2(\omega^2)^2 - 2km\omega^2 + \frac{k^2}{2} = 0\ , \tag{31}$$

whose solutions are

$$\omega_1^2 = \frac{k}{2m}(2+\sqrt{2})\ , \quad \omega_2^2 = \frac{k}{2m}(2-\sqrt{2})\ . \tag{32}$$

Method 2: Calculate eigenvalues of the matrix

$$\hat{m}^{-1}\hat{k} = \frac{k}{2m}\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \tag{33}$$

since the Euler-Lagrange equations with harmonic ansatz gives $(\hat{k}-\omega^2\hat{m})\vec{a} = \vec{0}$, which can be written as $(\hat{m}^{-1}\hat{k})\vec{a} = \omega^2\vec{a}$, for invertible mass matrix.

(e) 4 pt *Derive eigenvectors*

Method 1: Solve $(\hat{k} - \omega_{1,2}^2 \hat{m})\vec{a}_{1,2} = \vec{0}$ for constant vectors $\vec{a}_{1,2}$. For instance,

$$(\hat{k} - \omega_1^2 \hat{m})\vec{a}_1 = \begin{bmatrix} k - 2m\omega_1^2 & -m\omega_1^2 \\ -m\omega_1^2 & \frac{k}{2} - m\omega_1^2 \end{bmatrix} \begin{bmatrix} a_{1,1} \\ a_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

Solve the first equation, $(k - 2m\omega_1^2)a_{1,1} - m\omega_1^2 a_{1,2} = 0$,

$$\begin{aligned} a_{1,2} &= \frac{k - 2m\omega_1^2}{m\omega_1^2} a_{1,1} = \Big(\frac{k}{m\omega_1^2} - 2\Big) a_{1,1} = \Big(\frac{2}{2+\sqrt{2}} - 2\Big) a_{1,1} \\ &= \Big(\frac{2(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} - 2\Big) a_{1,1} = \Big((2-\sqrt{2}) - 2\Big) a_{1,1} = -\sqrt{2} a_{1,1} \end{aligned} \tag{35}$$

Thus $\vec{a}_1 = a_{1,1}(1, -\sqrt{2})$ is determined up to an overall constant. Similarly,

$$\vec{a}_1 = A_1 \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} , \quad \vec{a}_2 = A_2 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} , \tag{36}$$

with undetermined prefactors $A_{1,2}$.

Method 2: Compute the eigenvectors for the matrix $\hat{m}^{-1}\hat{k}$.

(f) 3 pt *Describe and sketch eigenmodes*

The eigenvalue ω_1 and eigenvector $\vec{a}_1$ correspond to the motion where the two masses move in opposite directions.

The eigenvalue ω_2 and eigenvector $\vec{a}_2$ correspond to the motion where the two masses move in the same direction.

(g) 4 pt *Explain how to find general solution*

Method 1: Decompose $\vec{q}$ in the basis of eigenvectors: $\vec{q} = \sum_{s=1}^2 r_s \vec{a}_s$. The eigenvectors in eq. (36) should be properly normalized:

$$\left.\begin{aligned} \vec{a}_s \cdot \hat{m} \cdot \vec{a}_{s'} &= \delta_{ss'} \\ \vec{a}_s \cdot \hat{k} \cdot \vec{a}_{s'} &= \omega_s^2 \delta_{ss'} \end{aligned}\right\} \quad \Big(\Rightarrow \quad A_s = \frac{\omega_s}{\sqrt{2k}}\Big) . \tag{37}$$

Then the Lagrangian is, in terms of the normal coordinates r_s, guaranteed to be diagonal and "canonically" normalized:

$$L = \sum_{s=1}^{2} L_s , \quad L_s = \tfrac{1}{2}\dot{r}_s^2 - \tfrac{1}{2}\omega_s^2 r_s^2 \;\Rightarrow\; r_s = C_s \cos(\omega_s t + \phi_s) . \tag{38}$$

Inserting this into $\vec{q} = \sum_{s=1}^2 r_s \vec{a}_s$ gives the final result for $\vec{q}$.

Method 2: Construct the matrix $\hat{A}$, whose columns are comprised of the two eigenvectors with the normalisation factors $A_{1,2}$ set to 1:

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ \sqrt{2} & -\sqrt{2} \end{bmatrix} . \tag{39}$$

Insert $\vec{q} = \hat{A} \cdot \vec{Q}$ into the Lagrangian, which diagonalises: $L = L_1 + L_2$, with

$$\begin{aligned} L_1 &= m(2-\sqrt{2})\dot{Q}_1^2 - kQ_1^2 , \\ L_2 &= m(2+\sqrt{2})\dot{Q}_2^2 - kQ_2^2 . \end{aligned} \tag{40}$$

Find the corresponding Euler-Lagrange equations,

$$m(2-\sqrt{2})\ddot{Q}_1 + kQ_1 = 0 \quad \Rightarrow \quad \ddot{Q}_1 + \omega_1^2 Q_1 = 0\ . \tag{41}$$

$$m(2+\sqrt{2})\ddot{Q}_2 + kQ_2 = 0 \quad \Rightarrow \quad \ddot{Q}_2 + \omega_2^2 Q_2 = 0\ . \tag{42}$$

Solve these equations by

$$Q_1 = C_1 \cos(\omega_1 t + \phi_1)\ , \quad Q_2 = C_2 \cos(\omega_2 t + \phi_2)\ . \tag{43}$$

Insert these solutions into $\vec{q} = \hat{A} \cdot \vec{Q}$ to find $\vec{q}$. Finally, use $q_1 = x$ and $q_2 = \ell\theta$ to obtain the general solution for x and θ.

(h) 4 pt *Imposing boundary conditions*
The conditions $x(0) = 0$ and $\theta(0) = 0$ produce

$$C_1 \sin(\phi_1) + C_2 \sin(\phi_2) = 0\ , \tag{44}$$

$$-C_1 \sin(\phi_1) + C_2 \sin(\phi_2) = 0\ , \tag{45}$$

which have the solution

$$\phi_1 = \phi_2 = 0\ . \tag{46}$$

The velocity conditions $\dot{x}(0) = 0$ and $\dot{\theta}(0) = v_0/\ell$ yield

$$C_1\omega_1 \cos(\phi_1) + C_2\omega_2 \cos(\phi_2) = 0\ , \tag{47}$$

$$\frac{\sqrt{2}}{\ell}(-C_1\omega_1 \cos(\phi_1) + C_2\omega_2 \cos(\phi_2)) = \frac{v_0}{\ell}\ , \tag{48}$$

Inserting $\phi_1 = \phi_2 = 0$, this becomes

$$C_1\omega_1 + C_2\omega_2 = 0\ , \tag{49}$$

$$\frac{\sqrt{2}}{\ell}(-C_1\omega_1 + C_2\omega_2) = \frac{v_0}{\ell}\ , \tag{50}$$

which have the solution

$$C_1 = \frac{-1}{2\sqrt{2}}\,\frac{v_0}{\omega_1}\ , \quad C_2 = \frac{1}{2\sqrt{2}}\,\frac{v_0}{\omega_2}\ . \tag{51}$$

Solution of exercise 4: HamiltonianMechanics

(a) 4 pt *Compute Hamiltonian*
First calculate the conjugate momentum

$$p \equiv \frac{\partial L}{\partial \dot{x}} = -mc^2 \frac{\partial}{\partial v}\sqrt{1 - \frac{v^2}{c^2}} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \ .$$

Note: this clearly gives the correct value mv in the non-relativistic limit.
Solve this for $v = \dot{x}$, in order to substitute that later into the Hamiltonian:

$$v = \frac{cp}{\sqrt{p^2 + m^2c^2}}$$

The Hamiltonian is then given by

$$H \equiv pv - L = pv + mc^2\sqrt{1 - \frac{v^2}{c^2}} = \frac{cp^2}{\sqrt{p^2 + m^2c^2}} + mc^2\sqrt{1 - \frac{p^2}{p^2 + m^2c^2}}$$
$$= c\sqrt{p^2 + m^2c^2}$$

(b) 5 pt *Calculate the Poisson brackets. Find $x(t)$*

$$\{p, H\} = \frac{\partial p}{\partial p}\frac{\partial H}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial H}{\partial p} = 0 \ , \quad \text{this means that } p \text{ is conserved!} \tag{52}$$
$$\{x, H\} = \frac{\partial x}{\partial p}\frac{\partial H}{\partial x} - \frac{\partial x}{\partial x}\frac{\partial H}{\partial p} = -\frac{\partial H}{\partial p} = \frac{-cp}{\sqrt{p^2 + m^2c^2}} \ . \tag{53}$$

Combining this with the Hamilton equation $\dot{x} = \frac{\partial H}{\partial p}$ we get

$$\dot{x} = \frac{cp}{\sqrt{p^2 + m^2c^2}} = \text{constant} \ . \tag{54}$$

Therefore

$$x(t) = \frac{cpt}{\sqrt{p^2 + m^2c^2}} \ . \tag{55}$$

(c) 3 pt *Derive the Hamilton equations of motion*
Note: From here on $H = H_0 + \lambda H_0^2$. Note: The parameter λ is not small.
The Hamilton equations are

$$\dot{x} = \frac{\partial H}{\partial p} = (1 + 2\lambda H_0)\frac{\partial H_0}{\partial p} = (1 + 2\lambda H_0)\frac{p}{m} \ ,$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -(1 + 2\lambda H_0)\frac{\partial H_0}{\partial x} = -(1 + 2\lambda H_0)m\omega_0^2 x \ . \tag{56}$$

(d) $\boxed{\text{3 pt}}$ *Show H_0 is conserved*
In general we have that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \ . \tag{57}$$

Inserting $f = H_0$ we find

$$\frac{dH_0}{dt} = \frac{\partial H_0}{\partial t} + \{H, H_0\} \ . \tag{58}$$

H_0 does not depend on time explicitly, so $\partial H_0/\partial t = 0$. The Poisson bracket is also zero:

$$\{H, H_0\} = \{H_0 + \lambda H_0^2, H_0\} = \{H_0, H_0\} + \lambda\{H_0^2, H_0\} = (1 + 2\lambda)\{H_0, H_0\} = 0 \tag{59}$$

due to antisymmetry of the Poisson bracket. We conclude that $dH_0/dt = 0$, in other words that H_0 is conserved.

(e) $\boxed{\text{3 pt}}$ *Show x satisfies harmonic differential equation*
Use the Hamilton equations in eq. (56) and the fact that H_0 is constant. Take the time derivative of the first equation and insert the second equation.

$$\ddot{x} = (1 + 2\lambda H_0)\frac{\dot{p}}{m} = -(1 + 2\lambda H_0)^2\omega_0^2 x \equiv -\omega^2 x \ , \tag{60}$$

where $\omega \equiv \omega_0(1 + 2\lambda H_0)$.

(f) $\boxed{\text{4 pt}}$ *Solve the differential equation. Re-express ω.*
The general solution is

$$x(t) = A\cos(\omega t + \phi) \ . \tag{61}$$

From this one can calculate p, upon inverting the first Hamilton equation,

$$p = \frac{m\dot{x}}{1 + 2\lambda H_0} = \frac{-m\omega A\sin(\omega t + \phi)}{1 + 2\lambda H_0} = -m\omega_0 A\sin(\omega t + \phi) \ . \tag{62}$$

Now one can calculate H_0

$$\begin{aligned} H_0 &= \frac{(-m\omega_0 A\sin(\omega t + \phi))^2}{2m} + \tfrac{1}{2}m\omega_0^2(A\cos(\omega t + \phi))^2 \\ &= \tfrac{1}{2}m\omega_0^2 A^2 \end{aligned} \tag{63}$$

Inserting this into the definition of ω yields

$$\omega = \omega_0(1 + \lambda m\omega_0^2 A^2) \ . \tag{64}$$

We see that the frequency receives anharmonic corrections that scale as the amplitude squared, as usual.