Solution of exercise 1: Questions

(a) [2 pt] *Which Lagrangians are equivalent?*
L_1 and L_3 are equivalent to L, but L_2 is not.

(b) [2 pt] *Incorporate constraint*
• Method 1: Eliminate one of the three spacial coordinates from the Lagrangian, e.g. by using spherical coordinates ($r = R, \theta, \phi$).

• Method 2: Include a Lagrange multiplier term $L \to L + \lambda f(x, y, z)$, with $f(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0$.

(c) [1 pt] *Invariance under spacial rotations*
The angular momentum is conserved.

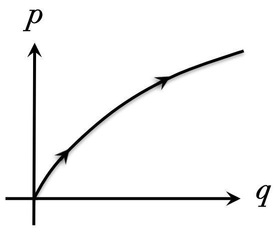
(d) [1 pt] *Virial theorem*

$$\langle T \rangle = \frac{n}{2} \langle U \rangle$$

(e) [1 pt] *Value of the total cross section for Rutherford scattering*
The total cross section is infinite (due to contributions from small angle scattering)

(f) [1 pt] *Property Poisson bracket*
The Poisson bracket $\{I_1, I_2\}$ is also time independent.

(g) [1 pt] *Path in phase space*



(h) [1 pt] *Number of* different *principal moments of inertia*
A symmetric top has 2 different principle moments of inertia: (I, I, I_3).

(i) [1 pt] *Bracket after canonical transformation*
A set of coordinates $\vec{q}$ and their conjugate momenta $\vec{p}$ satisfy by definition the Poisson bracket relation $\{q_i, p_j\} = \delta_{ij}$.
The Poisson bracket is conserved under canonical transformations. Hence,

$$\{Q_i, P_j\} = \delta_{ij} \ . \tag{4}$$

(j) [3 pt] *Moment of inertia thin rod*
The mass is $m = \mu\ell$. The moment of inertia around axes through middle is

$$I = \mu \int_{-\ell/2}^{\ell/2} dr\ r^2 = \mu \frac{r^3}{3} \Big|_{r=-\ell/2}^{r=\ell/2} = \frac{\mu\ell^3}{12} = \frac{m\ell^2}{12} \ . \tag{5}$$

(k) $\boxed{\text{4 pt}}$ *Oscillator with friction*

Inserting the ansatz into the differential equation gives the equation

$$-\lambda^2 + 2i\kappa\lambda + \omega^2 = 0 \quad \Rightarrow \quad \lambda_\pm = i\kappa \pm \sqrt{\omega^2 - \kappa^2} \equiv i\kappa \pm \omega_0 \ . \tag{6}$$

The general solution is

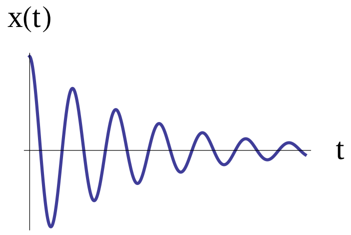
$$x = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t} \tag{7}$$

$$= A_+ e^{-\kappa t} e^{i\omega_0 t} + A_- e^{-\kappa t} e^{-i\omega_0 t} \ . \tag{8}$$

Moreover x must be real, so we get

$$x = A e^{-\kappa t} \cos(\omega_0 t) \ . \tag{9}$$

The sketch is



(l) $\boxed{\text{4 pt}}$ *Two-particle system*

The symmetry transformation $x_i \to x_i + \epsilon$ leads to conservation of the center-of-mass momentum $(m_{\text{tot}}\dot{R})$.

To rewrite the Lagrangian, first solve the definitions $R = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$ and $r = x_1 - x_2$ for x_1 and x_2:

$$x_1 = R + \frac{m_2}{m_1 + m_2} r \ , \tag{10}$$

$$x_2 = R - \frac{m_1}{m_1 + m_2} r \ . \tag{11}$$

Inserting this into the Lagrangian gives

$$L = \frac{m_1 + m_2}{2} \dot{R}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{r}^2 - U(r) \tag{12}$$

The motion of the center-of-mass is trivial and one can focus on the equations of motion for the relative position r with associated *reduced mass* $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

Solution of exercise 2: KeplerProblem

(a) 4 pt *Demonstrate reduction to one-dimensional problem*
The Lagrangian has a central potential and is thus symmetric under rotations $r_i \to r_i + \epsilon\, \epsilon_{ijk} n_j r_k$. Hence, angular momentum $\vec{M}$ is conserved.

From the expression $\vec{M} = \vec{r} \times \vec{p}$ we see that $\vec{r}$ is always perpendicular to the constant vector $\vec{M}$. If we pick the coordinate system such that $\vec{M}$ points along the z-axis, then $\vec{r}$ is confined to the two-dimensional x, y-plane. Thus $z = 0$.

With polar coordinates (r, θ) in the x, y-plane, the Lagrangian reads

$$L = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r) \ . \qquad (13)$$

Using the expression $M = mr^2\dot{\theta}$, the energy of the particle can be written in terms of the coordinate r only:

$$E = \tfrac{1}{2} m \dot{r}^2 + \frac{M^2}{2mr^2} + U(r) \ . \qquad (14)$$

This can be solved for either t or θ as an integral over r:

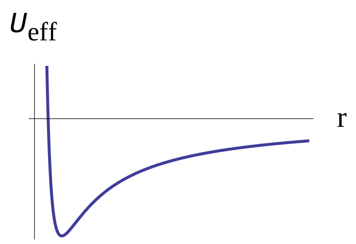
$$t = \int dr \frac{1}{\sqrt{(2/m)(E - U(r)) - M^2/(m^2 r^2)}} \ , \qquad (15)$$

$$\theta = \int dr \frac{M/r^2}{\sqrt{2m(E - U(r)) - M^2/r^2}} \ , \qquad (16)$$

(b) 5 pt *Give and sketch $U_{\text{eff}}(r)$. Discuss orbits.*
From eq. (14) we see that

$$U_{\text{eff}}(r) = \frac{M^2}{2mr^2} + U(r) \ . \qquad (17)$$

With $U(r) = -k/r$, we have a negative $1/r$ plus a positive $1/r^2$. For large values of r the $1/r$ dominates, so the potential is negative there. For small values of r the $1/r^2$ dominates and the potential goes to $+\infty$. The sketch is



The orbits are:

(i) $E = U_{\text{min}} < 0$: circular orbit (closed)
(ii) $U_{\text{min}} < E < 0$: elliptic orbit (closed)
(iii) $E > 0$: hyperbolic orbit (open)

(c) [5 pt] *Compute $\{H, A_i\}$ and give interpretation*

The Runge-Lenz vector is conserved if $\{H, A_i\} = 0$ (since A_i is not explicitly time-dependent). So let's show that:

$$\begin{aligned}\{H, A_i\} &= \left\{\frac{p^2}{2m} - \frac{k}{r}\ ,\ \epsilon_{ijk} p_j M_k - mk\frac{r_i}{r}\right\} \\ &= \frac{\epsilon_{ijk}}{2m}\left\{p^2, p_j M_k\right\} - \frac{k}{2}\left\{p^2, \frac{r_i}{r}\right\} - k\epsilon_{ijk}\left\{\frac{1}{r}, p_j M_k\right\} + mk^2\left\{\frac{1}{r}, \frac{r_i}{r}\right\}\end{aligned} \tag{18}$$

Calculate the four brackets in turn:

$$\begin{aligned}&\{p^2, p_j M_k\} = p_j\{p^2, M_k\} = 2p_j p_\ell\{p_\ell, M_k\} = 2p_j p_\ell \epsilon_{k\ell n} p_n = 0\ , \\ &\left\{p^2, \frac{r_i}{r}\right\} = 2p_k\left\{p_k, \frac{r_i}{r}\right\} = \frac{2p_k}{r}\left\{p_k, r_i\right\} + 2p_k r_i\left\{p_k, \frac{1}{r}\right\} = \frac{2p_i}{r} - \frac{2r_i r_k p_k}{r^3}\ , \\ &\left\{\frac{1}{r}, p_j M_k\right\} = p_j\left\{\frac{1}{r}, M_k\right\} + M_k\left\{\frac{1}{r}, p_j\right\} = p_j\left\{\frac{1}{r}, \epsilon_{k\ell m} r_\ell p_m\right\} + M_k\frac{1}{r^2}\frac{r_j}{r} \\ &\qquad = p_j \epsilon_{k\ell m} r_\ell\left\{\frac{1}{r}, p_m\right\} + M_k\frac{1}{r^2}\frac{r_j}{r} = M_k\frac{1}{r^2}\frac{r_j}{r} = \frac{r_j M_k}{r^3}\ , \\ &\left\{\frac{1}{r}, \frac{r_i}{r}\right\} = 0\ .\end{aligned}$$

Inserting these results gives

$$\begin{aligned}\{H, A_i\} &= -\frac{k}{2}\left(\frac{2p_i}{r} - \frac{2r_i r_k p_k}{r^3}\right) - k\epsilon_{ijk}\frac{r_j M_k}{r^3} \\ &= -k\left(\frac{p_i}{r} - \frac{r_i r_k p_k}{r^3}\right) - \frac{k}{r^3}\epsilon_{ijk} r_j \epsilon_{k\ell m} r_\ell p_m \\ &= -k\left(\frac{p_i}{r} - \frac{r_i r_k p_k}{r^3}\right) - \frac{k}{r^3}\left(r_i r_k p_k - r_k r_k p_i\right) = 0\ .\end{aligned} \tag{19}$$

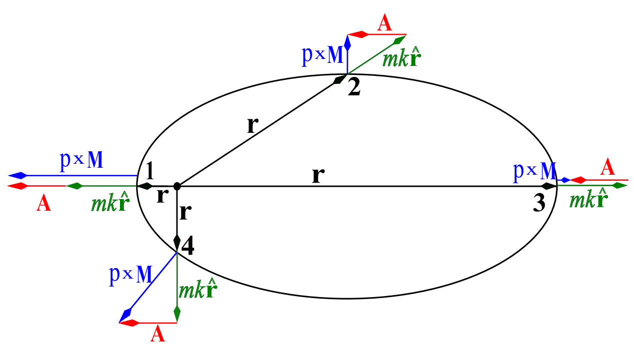
(d) [2 pt] *Argue RL-vector lies in orbital plane*

The Runge-Lenz vector is

$$\vec{A} = \vec{p} \times \vec{M} - \frac{mk}{r}\vec{r}\ . \tag{20}$$

The second term is proportional to $\vec{r}$ and therefore obviously lies in the plane of the orbit. The first term is perpendicular to $\vec{M}$. Also the orbital plane is perpendicular to $\vec{M}$. Hence, the first term is also in the orbital plane. Conclusion is that $\vec{A}$ lies in the orbital plane.

(e) [4 pt] *Sketch sun, planet, elliptic orbit, and RL-vector at various locations*



The first term of the RL vector, $\vec{p}\times\vec{M}$, points towards the outside of the orbit. Contrary, the second term, $-mk\hat{r}$, points towards the inside of the orbit. At the perihelion (point 1) and aphelion (point 3), these two vectors are paralel. At the perihelion, $\vec{p}$ is large and $\vec{r}$ is small, so the first term dominates and the vector sum is in the direction from aphelion to perihelion.

(f) 6 pt *Determine the shape of the orbit* $r(\phi)$ *and eccentricity* ϵ
Compute $\vec{A}\cdot\vec{r}$ in two ways:

$$\vec{A}\cdot\vec{r} = Ar\cos(\phi) \ , \tag{21}$$

$$\vec{A}\cdot\vec{r} = \left(\vec{p}\times\vec{M} - \frac{mk}{r}\vec{r}\right)\cdot\vec{r} = M^2 - mkr \ , \tag{22}$$

where we used the cyclic symmetry of the triple product $(\vec{p}\times\vec{M})\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{M} = M^2$. Hence,

$$Ar\cos(\phi) = M^2 - mkr \implies r = \frac{M^2}{mk + A\cos(\phi)} = \frac{\frac{M^2}{mk}}{1+\frac{A}{mk}\cos(\phi)} \ . \tag{23}$$

Comparing this to the general formula $r = r_0/(1+\epsilon\cos(\phi))$ we identify

$$\epsilon = \frac{A}{mk} \ . \tag{24}$$

Solution of exercise 3: CoupledPendulums

(a) [4 pt] *Construct Lagrangian*

Let the origin of the coordinate system coincide with the equilibrium position of mass m_1. Then the positions of the masses m_1 and m_2 are, respectively

$$(x_1, y_1) = (\ell \sin\theta_1, \ell(1 - \cos\theta_1)) \ , \tag{25}$$
$$(x_2, y_2) = (x + \ell \sin\theta_2, \ell(1 - \cos\theta_2)) \ . \tag{26}$$

The two pendulums thus have kinetic and potential energies

$$T_i = \tfrac{1}{2} m_i \ell^2 \dot\theta_i^2 \ , \quad U_i = m_i g \ell (1 - \cos\theta_i) \ . \tag{27}$$

The spring has potential energy

$$U_{\text{spring}} = k(d - x)^2 \ , \tag{28}$$

where d is the length of the spring

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= \sqrt{(x + \ell\sin\theta_2 - \ell\sin\theta_1)^2 + (\ell\cos\theta_1 - \ell\cos\theta_2)^2} \end{aligned} \tag{29}$$

The Lagrangian is

$$L = T_1 + T_2 - U_1 - U_2 - U_{\text{spring}} \tag{30}$$

(b) [4 pt] *Construct E-L equations*

$$m_i \ell^2 \ddot\theta_i = -m_i g \ell \sin\theta_i - 2k(d - x)\frac{\partial d}{\partial \theta_i} \qquad \text{for } i = 1, 2 \tag{31}$$

with

$$\begin{aligned} \frac{\partial d}{\partial \theta_1} &= \frac{1}{d}\big[(x + \ell\sin\theta_2 - \ell\sin\theta_1)(-\ell\cos\theta_1) + (\ell\cos\theta_1 - \ell\cos\theta_2)(-\ell\sin\theta_1)\big] \\ &= \frac{1}{d}\big[-\ell x\cos\theta_1 - \ell^2\sin\theta_2\cos\theta_1 + \ell^2\sin\theta_1\cos\theta_2\big] \\ &= \frac{\ell}{d}\big[-x\cos\theta_1 + \ell\sin(\theta_1 - \theta_2)\big] \ , \end{aligned} \tag{32}$$

Similarly,

$$\frac{\partial d}{\partial \theta_2} = \frac{\ell}{d}\big[x\cos\theta_2 - \ell\sin(\theta_1 - \theta_2)\big] \ . \tag{33}$$

(c) [4 pt] *Approximating Lagrangian*
When the angles are small, $d \approx x + \ell(\theta_2 - \theta_1)$, so

$$U_{\text{spring}} \approx k\ell^2(\theta_2 - \theta_1)^2 \ , \quad U_i \approx \tfrac{1}{2} m_i g \ell \theta_i^2 \ . \tag{34}$$

(d) $\boxed{4\text{ pt}}$ *Eigenfrequencies*

$$\omega_1^2 = \frac{g}{\ell} \;, \quad \text{or} \;\; \omega_2^2 = \frac{g}{\ell}(1+2\eta) \;. \tag{35}$$

With $\eta = 2k\ell/(mg)$ the eigenfrequencies may also be written as

$$\omega_\pm^2 = \frac{g}{\ell} + \frac{k}{m}(2 \pm 2) \;. \tag{36}$$

(e) $\boxed{4\text{ pt}}$ *Eigenvectors*

$$\vec{a}_1 = A_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \;, \quad \vec{a}_2 = A_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \;. \tag{37}$$

(f) $\boxed{3\text{ pt}}$ *Sketch eigenmodes*
Even without the explicit calculation of eigenvectors, one can easily guess that the modes are:
Pendulums move in phase (same direction) $\Leftrightarrow$ $\omega_1^2 = g/\ell$
Pendulums move out of phase (opposite directions) $\Leftrightarrow$ $\omega_2^2 = g/\ell(1+2\eta)$

(g) $\boxed{3\text{ pt}}$ *Explain how to find general solution*
• Method 1: Decompose $\vec{q}$ in the basis of eigenvectors: $\vec{q} = \sum_{s=1}^{2} r_s \vec{a}_s$. The eigenvectors should be properly normalized:

$$\begin{aligned} \vec{a}_s \cdot \hat{m} \cdot \vec{a}_{s'} &= \delta_{ss'} \\ \vec{a}_s \cdot \hat{k} \cdot \vec{a}_{s'} &= \omega_s^2 \delta_{ss'} \end{aligned} \tag{38}$$

Then the Lagrangian is, in terms of the normal coordinates r_s, guaranteed to be diagonal and "canonically" normalized:

$$L = \sum_{s=1}^{2} L_s \;, \quad L_s = \tfrac{1}{2}\dot{r}_s^2 - \tfrac{1}{2}\omega_s^2 r_s^2 \;\Rightarrow\; r_s = C_s \cos(\omega_s t + \phi_s) \;. \tag{39}$$

Inserting this into $\vec{q} = \sum_{s=1}^{2} r_s \vec{a}_s$ gives the final result for $\vec{q}$.
• Method 2: Construct the matrix $\hat{A}$, whose columns are comprised of the two eigenvectors with the normalisation factors $A_{1,2}$ set to 1:

$$\hat{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \;. \tag{40}$$

Insert $\vec{q} = \hat{A} \cdot \vec{Q}$ into the Lagrangian, which diagonalises: $L = L_1 + L_2$, but the L_i are not necessarily "canonically" normalized. Find the corresponding Euler-Lagrange equations, and solve them. Insert the solutions into $\vec{q} = \hat{A} \cdot \vec{Q}$ to find $\vec{q}$.

Solution of exercise 4: Hamiltonian

(a) 4 pt *Derive Hamiltonian equations*
It is useful to write the Hamiltonian in index notation

$$H = \frac{p_k^2}{2m} - \gamma \epsilon_{k\ell m} B_k r_\ell p_m \tag{41}$$

The Hamilton equations are then

$$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} - \gamma \epsilon_{k\ell i} B_k r_\ell = \frac{p_i}{m} + \gamma(\vec{r} \times \vec{B})_i \ , \tag{42}$$

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = \gamma \epsilon_{kim} B_k p_m = \gamma(\vec{p} \times \vec{B})_i \tag{43}$$

(b) 6 pt *Compute time derivative of* $\vec{M}$
$\dot{\vec{M}}$ can be computed with the EOM or with Poisson brackets.
• With EOM:

$$\begin{aligned} \dot{\vec{M}} &= \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \\ &= \left(\frac{\vec{p}}{m} + \gamma(\vec{r} \times \vec{B})\right) \times \vec{p} + \vec{r} \times \left(\gamma(\vec{p} \times \vec{B})\right) \\ &= \gamma \left[(\vec{r} \times \vec{B}) \times \vec{p} + \vec{r} \times (\vec{p} \times \vec{B})\right] \\ &= \gamma \left[(\vec{r} \times \vec{p}) \times \vec{B}\right] = \gamma \vec{M} \times \vec{B} \ . \end{aligned} \tag{44}$$

Here we used the Jacobi identity for the cross product,

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{c} \times (\vec{a} \times \vec{b}) + \vec{b} \times (\vec{c} \times \vec{a}) = 0 \ ,$$

Alternatively, we could have written out the cross products in index notation:

$$\begin{aligned} [(\vec{r} \times \vec{B}) \times \vec{p}]_i &= \epsilon_{ijk} (\vec{r} \times \vec{B})_j p_k \\ &= \epsilon_{ijk} \epsilon_{j\ell m} r_\ell B_m p_k \\ &= \epsilon_{jki} \epsilon_{j\ell m} r_\ell B_m p_k \\ &= (\delta_{k\ell}\delta_{im} - \delta_{km}\delta_{i\ell}) r_\ell B_m p_k \\ &= (\vec{r} \cdot \vec{p}) B_i - (\vec{p} \cdot \vec{B}) r_i \ , \end{aligned} \tag{45}$$

and similarly

$$[\vec{r} \times (\vec{p} \times \vec{B})]_i = (\vec{r} \cdot \vec{B}) p_i - (\vec{r} \cdot \vec{p}) B_i \ . \tag{46}$$

Adding the two terms gives what we claimed:

$$\begin{aligned} [(\vec{r} \times \vec{B}) \times \vec{p} + \vec{r} \times (\vec{p} \times \vec{B})]_i &= (\vec{r} \cdot \vec{B}) p_i - (\vec{p} \cdot \vec{B}) r_i \\ &= (\delta_{i\ell}\delta_{mk} - \delta_{ik}\delta_{m\ell}) r_k p_\ell B_m \\ &= \epsilon_{xim} \epsilon_{x\ell k} r_k p_\ell B_m \\ &= \epsilon_{xim} (\vec{p} \times \vec{r})_x B_m \\ &= [(\vec{r} \times \vec{p}) \times \vec{B}]_i \ . \end{aligned} \tag{47}$$

- With Poisson brackets:

$$\dot{\vec{M}} = \frac{d\vec{M}}{dt} = \frac{\partial \vec{M}}{\partial t} + \{H, \vec{M}\} = \{H, \vec{M}\} \ . \tag{48}$$

The Poisson bracket is

$$\{H, M_i\} = \left\{ \frac{p_k^2}{2m} - \gamma B_k M_k \, , \, M_i \right\} = \frac{p_k}{m}\{p_k \, , \, M_i\} - \gamma B_k \{M_k \, , \, M_i\} \ . \tag{49}$$

Calculating and inserting the following results

$$\{M_i, p_j\} = -\epsilon_{ijk} p_k \ , \quad \{M_i, M_j\} = -\epsilon_{ijk} M_k \ , \tag{50}$$

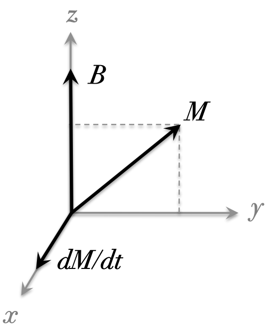
gives

$$\{H, M_i\} = \frac{p_k}{m}\big(\epsilon_{ik\ell} p_\ell\big) - \gamma B_k \big(\epsilon_{ik\ell} M_\ell\big) = \gamma(\vec{M} \times \vec{B})_i \ . \tag{51}$$

Thus

$$\dot{\vec{M}} = \gamma \vec{M} \times \vec{B} \ . \tag{52}$$

(c) 4 pt *Sketch the vectors and argue for precession*
Sketch must have $\dot{\vec{M}}$ along the positive x-axis (assuming $\gamma > 0$).



Argue that $\vec{M}$ precesses around B by noting that
(i) The length $|\vec{M}|$ is constant ($\dot{\vec{M}}$ is perpendicular to $\vec{M}$); and
(ii) The component of $\vec{M}$ along $\vec{B}$ is constant ($\dot{\vec{M}}$ is perpendicular to $\vec{B}$). Therefore, in general $\vec{M}$ will rotate around $\vec{B}$.

(d) 6 pt *Solve for* $\vec{M}$
The equations are

$$\dot{M}_1 = -\gamma B \, M_2 \ , \tag{53}$$
$$\dot{M}_2 = +\gamma B \, M_1 \ , \tag{54}$$
$$\dot{M}_3 = 0 \ . \tag{55}$$

Decoupling the differential equations gives $\ddot{M}_i = -(\gamma B)^2 M_i \equiv -\omega^2 M_i$ for $i = 1, 2$. The solution is, after imposing the boundary condition,

$$M_1 = (M_0)_1 \cos(\omega t) + (M_0)_2 \sin(\omega t) \ , \tag{56}$$
$$M_1 = -(M_0)_1 \sin(\omega t) + (M_0)_2 \cos(\omega t) \ , \tag{57}$$
$$M_3 = (M_0)_3 \ . \tag{58}$$

(e) 6 pt *Construct Lagrangian and EOM*

With $\vec{B}$ along the z-axis, the Hamiltonian becomes

$$H = \frac{p^2}{2m} - \gamma B M_3 \ . \tag{59}$$

The Lagrangian is then

$$\begin{aligned} L &= \vec{p}\cdot\dot{\vec{r}} - H \\ &= \vec{p}\cdot\dot{\vec{r}} - \frac{p^2}{2m} + \gamma B M_3 \end{aligned} \tag{60}$$

We must substitute the canonical momentum in terms of $\dot{\vec{r}}$. Using the first Hamilton equation we find

$$\vec{p} = m\big[\dot{\vec{r}} + \gamma \vec{B}\times\vec{r}\big] = m\big[\dot{\vec{r}} + \gamma B(r_1\hat{e}_2 - r_2\hat{e}_1)\big] = m\begin{bmatrix}\dot{r}_1 - \gamma B r_2 \\ \dot{r}_2 + \gamma B r_1 \\ \dot{r}_3\end{bmatrix} \ . \tag{61}$$

With this the three terms in the Lagrangian become

$$\vec{p}\cdot\dot{\vec{r}} = m\dot{r}^2 + m\gamma B(r_1\dot{r}_2 - r_2\dot{r}_1) \ , \tag{62}$$

$$-\frac{p^2}{2m} = -\frac{m}{2}\dot{r}^2 - \frac{m}{2}(\gamma B)^2(r_1^2 + r_2^2) - m\gamma B(r_1\dot{r}_2 - r_2\dot{r}_1) \ , \tag{63}$$

$$\gamma B M_3 = m\gamma B(r_1\dot{r}_2 - r_2\dot{r}_1) + m(\gamma B)^2(r_1^2 + r_2^2) \ . \tag{64}$$

Adding these terms up gives

$$L = \frac{1}{2}m\dot{r}^2 + m\gamma B(r_1\dot{r}_2 - r_2\dot{r}_1) + \frac{m}{2}(\gamma B)^2(r_1^2 + r_2^2) \ . \tag{65}$$

The Euler-Lagrange equations are

$$\ddot{r}_1 - \gamma B\dot{r}_2 = \gamma B\dot{r}_2 + \gamma^2 B^2 r_1 \ , \tag{66}$$

$$\ddot{r}_2 - \gamma B\dot{r}_1 = \gamma B\dot{r}_1 + \gamma^2 B^2 r_2 \ , \tag{67}$$

$$\ddot{r}_3 = 0 \ . \tag{68}$$