Lösung der Aufgabe 1

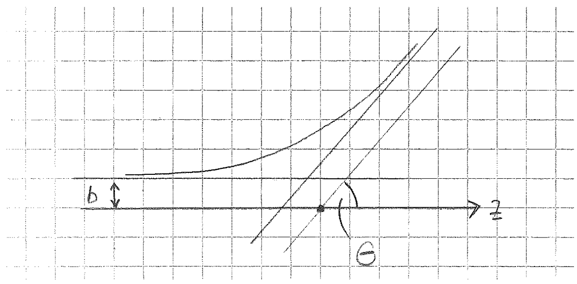
(a) 1 pt The EL equation immediately gives $m\ddot{x} = -\frac{dU(x)}{dx}$ (1 pt).

(b) 1 pt Add a total time derivative of an arbitrary function. Or multiply by an overall constant. Either of these two examples: (1 pt).

(c) 2 pt The components p_x, p_y (1 pt) and M_z (1 pt) are conserved.

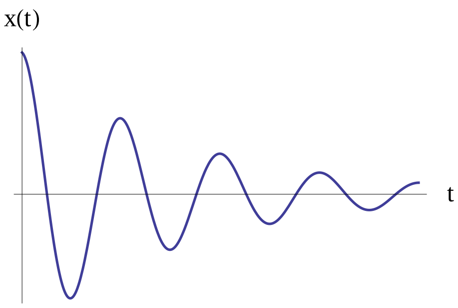
(d) 2 pt Due to rotational symmetry, angular momentum $\vec{M}$ is constant (1 pt). Since $\vec{M} = \vec{r}(t) \times \vec{p}(t)$, it holds that $\vec{M} \cdot \vec{r}(t) = 0$ for all times t. Hence $\vec{r}(t)$ lies in the plane perpendicular to $\vec{M}$ (1 pt).

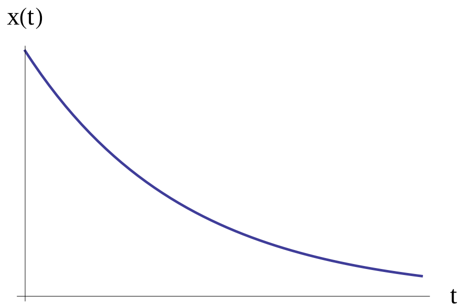
(e) 2 pt



(f) 2 pt We get resonance, so the amplitude will increase linearly in time (1 pt). In the presence of friction, this does not happen and the amplitude remains finite (1 pt).

(g) 2 pt Weak friction $\kappa < \omega$ (1 pt) Strong friction $\kappa > \omega$ (1 pt)

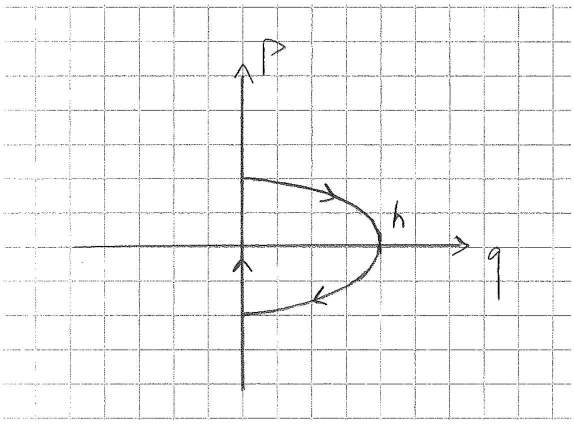




(h) 1 pt The argument uses Poisson brackets: (a) the Poisson bracket of two components of angular momentum yields the third component (given); and (b) the bracket of two conserved quantities is itself a conserved quantity (from Jacobi identity) (1 pt).

(i) 1 pt The Hamilton function is not invariant under canonical Transformations (1 pt).

(j) 2 pt



Lösung der Aufgabe 2

(a) 3 pt *Write down Lagrangian*
We start as usual with Cartesian coordinates:

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

We choose cylindrical coordinates, such that

$$\begin{aligned} x &= r\sin\alpha\cos\varphi \\ y &= r\sin\alpha\sin\varphi \\ z &= r\cos\alpha \end{aligned} \quad , \quad \begin{aligned} \dot{x} &= \dot{r}\sin\alpha\cos\varphi - r\sin\alpha\sin\varphi\dot{\varphi} \\ \dot{y} &= \dot{r}\sin\alpha\sin\varphi + r\sin\alpha\cos\varphi\dot{\varphi} \\ \dot{z} &= \dot{r}\cos\alpha \end{aligned}$$

where we took account of the constraint

$$\cos\alpha = \frac{z}{r}$$

in order to reduce the number of independent coordinates to two, which we chose to be r and φ. The Lagrangian is then

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\sin^2\alpha\dot{\varphi}^2\right) - mgr\cos\alpha \quad (1\,\text{pt})$$

We have

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}} &= m\dot{r} & \frac{\partial L}{\partial r} &= mr\sin^2\alpha\dot{\varphi}^2 - mg\cos\alpha \\ \frac{\partial L}{\partial \dot{\varphi}} &= mr^2\sin^2\alpha\dot{\varphi} & \frac{\partial L}{\partial \varphi} &= 0 \end{aligned} \quad ,$$

such that the Euler-Lagrange equations yield

$$\begin{aligned} \ddot{r} - r\sin^2\alpha\dot{\varphi}^2 + g\cos\alpha &= 0 \quad (1\,\text{pt}) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\sin^2\alpha\dot{\varphi}\right) &= 0 \quad (1\,\text{pt}) \end{aligned}$$

(b) 2 pt *Find conserved quantities*
We see immediately that L does not depend on time, such that the energy $E = T + V$ is conserved. It is thus given by

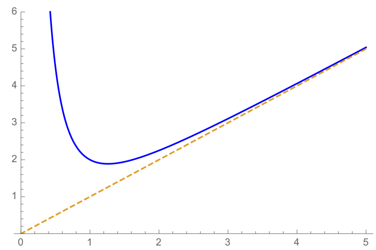
$$E = \frac{\partial L}{\partial \dot{r}}\dot{r} + \frac{\partial L}{\partial \dot{\varphi}}\dot{\varphi} - L = \frac{m}{2}\left(\dot{r}^2 + r^2\sin^2\alpha\dot{\varphi}^2\right) + mgr\cos\alpha \quad (1\,\text{pt}).$$

We also see that the coordinate φ is cyclic, which gives another conserved quantity

$$K = \frac{\partial L}{\partial \dot{\varphi}} = mr^2 \sin^2 \alpha \dot{\varphi}.$$

This comes from the invariance of L under rotations around the z-axis. The conserved quantity is the z-component of the angular momentum (1 pt).

(c) 2 pt *Reduce to 1D motion*
Since K is conserved, we can use it to eliminate $\dot{\varphi} = \frac{K}{mr^2 \sin^2 \alpha}$. The motion has then only one degree of freedom, r (1 pt). We can then write down the energy,



$$\begin{aligned} E &= \frac{m}{2}\dot{r}^2 + \frac{K^2}{2mr^2 \sin^2 \alpha} + mgr\cos\alpha \\ &= \frac{m}{2}\dot{r}^2 + U_{\text{eff}}(r), \end{aligned}$$

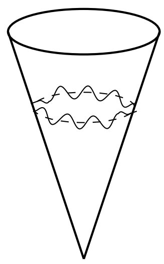
where we introduced the effective potential

$$U_{\text{eff}}(r) = mgr\cos\alpha + \frac{K^2}{2mr^2 \sin^2 \alpha} \qquad (1\,\text{pt}).$$

(d) 2 pt *Find circular trajectory*
We see that the effective potential has a minimum at some r_0. If the total energy of the particle happens to be just $E = U_{\text{eff}}(r_0)$, there is no kinetic energy remaining for motion in r. The particle then stays on a trajectory given by $r = r_0$ (1 pt). In order to determine r_0, we minimise the effective potential,

$$\begin{aligned} 0 &\stackrel{!}{=} \left.\frac{\partial U_{\text{eff}}}{\partial r}\right|_{r=r_0} = mg\cos\alpha - \frac{K^2}{mr_0^3 \sin^2 \alpha} \\ &\Rightarrow r_0 = \left(\frac{K^2}{m^2 g \sin^2 \alpha \cos\alpha}\right)^{\frac{1}{3}} \qquad (1\,\text{pt}). \end{aligned}$$

(e) 4 pt *Find oscillatory trajectory*
Going back to the effective potential, we see that close to r_0 the potential looks like a parabola. By performing an expansion in small deviations from r_0, we therefore expect the motion in r to be that of a harmonic oscillator (1 pt). In the full picture, the trajectory has also an angular component, such that the trajectory is given by a wavy line roughly centered around the circular trajectory from the last subquestion (1 pt). We determine the frequency of this oscillation by Taylor expanding the equations of motion. These are given as



$$\begin{aligned} 0 &= \ddot{r} - \frac{K^2}{m^2 r^3 \sin^2 \alpha} + g\cos\alpha \\ &= \ddot{\xi} - \frac{K^2}{m^2 (r_0 + \xi)^3 \sin^2 \alpha} + g\cos\alpha \\ &= \ddot{\xi} - \frac{K^2}{m^2 r_0^3 \sin^2 \alpha} + \frac{3K^2}{m^2 r_0^4 \sin^2 \alpha}\xi + g\cos\alpha + \mathcal{O}(\xi^2) \\ &= \ddot{\xi} + \frac{3K^2}{m^2 r_0^4 \sin^2 \alpha}\xi + \mathcal{O}(\xi^2) \qquad (1\,\text{pt}). \end{aligned}$$

We thus obtain as expected the equation of motion of a harmonic oscillator,

$$\ddot{\xi} + \omega^2 \xi = 0,$$

with

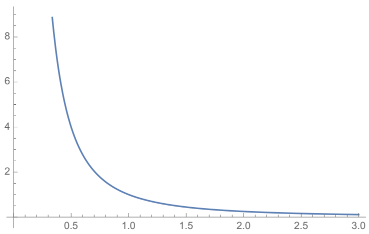
$$\omega^2 = \frac{3K^2}{m^2 r_0^4 \sin^2\alpha} = \frac{3g\cos\alpha}{r_0} \quad (1\,\text{pt}).$$

(f) 3 pt *Characterise motion*

Since gravity has been switched off, the effective potential is given solely by

$$U_{\text{eff}} = \frac{K^2}{2mr^2\sin^2\alpha}.$$

The motion in this effective potential is unbounded with a turning point when the effective potential becomes as big as the total energy (1 pt). It is possible for the particle to reach the tip of the cone, namely when the conserved quantity KS is zero, $\dot{\varphi} = 0$. Then the effective potential is also zero and will not prevent the particle from reaching the point $r = 0$ (1 pt).



(1 pt)

(g) 3 pt *Find angular deviation*

By using the chain rule, we get

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{K}{mr^2\sin^2\alpha}$$

$$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{mr^2\sin^2\alpha}{K}\dot{r} \quad (1\,\text{pt}).$$

We then use the fact that the energy is conserved to find an expression for $\dot{r}$:

$$E = \frac{m}{2}\dot{r}^2 + U_{\text{eff}}(r) \Rightarrow \dot{r} = \sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))} \quad (1\,\text{pt}).$$

Putting this together and integrating, we obtain

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{K}{m\sin^2\alpha}\frac{1}{r^2\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}} \Rightarrow \Delta\varphi = \frac{K}{m\sin^2\alpha}\int_{r_A}^{r_B}\frac{\mathrm{d}r}{r^2\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}}$$

$$= \frac{K}{m\sin^2\alpha}\int_{r_A}^{r_B}\frac{\mathrm{d}r}{r^2\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}}. \quad (1\,\text{pt}).$$

Note the presence of the factor $\frac{1}{\sin^2\alpha}$, which was not there for the motion in a central field.

(h) 3 pt *Find total deviation*

From symmetry, it is clear that the total angular difference will be equal to twice the difference from going from $r = r_{\text{min}}$ to $r = \infty$. We first determine r_{min}. It is the radius at which the effective potential becomes as big as the total energy

$$E = \frac{K^2}{2mr_{\text{min}}^2\sin^2\alpha} \Rightarrow r_{\text{min}} = \frac{K}{\sqrt{2mE}\sin\alpha}.$$

The above is worth (1 pt), altough not explicitly asked for. Alternatively, in the calculation below the students can use that $\left(Er_{\text{min}}^2 - \frac{K^2}{2m\sin^2\alpha}\right)$ is zero to evaluate the lower integral boundary, they then get the point for this.
With this we can evaluate the integral

$$\begin{aligned}
\frac{\Delta\varphi}{2} &= \frac{K}{m\sin^2\alpha}\int_{r_{\text{min}}}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{\frac{2}{m}(E - \frac{K^2}{2mr^2\sin^2\alpha})}} \\
&= -\frac{K}{m\sin^2\alpha}\int_{\frac{1}{r_{\text{min}}^2}}^{0} \frac{\mathrm{d}z}{\sqrt{\frac{2}{m}(E - \frac{K^2z^2}{2m\sin^2\alpha})}} \\
&= -\frac{K}{m\sin^2\alpha}\frac{1}{\frac{K}{m\sin\alpha}}\arcsin\left(\frac{\frac{Kz}{m\sin\alpha}}{\sqrt{\frac{2E}{m}}}\right)\Bigg|_{\frac{1}{r_{\text{min}}^2}}^{0} \\
&= -\frac{1}{\sin\alpha}(\arcsin(0) - \arcsin(1)) \\
&= \frac{\pi}{2\sin\alpha} \quad \text{(1 pt)},
\end{aligned}$$

where we have performed the substitution $z = \frac{1}{r}$ and used the integral in the formulae section. The total angular difference is thus

$$\Delta\varphi = \frac{\pi}{\sin\alpha}.$$

For $\alpha = \frac{\pi}{2}$, the cone becomes a plane, and the angular difference becomes $\Delta\varphi = \pi$, which is what we expect asymptotically from a straight line. For $\alpha \to 0$, the cone becomes very narrow, and we expect the particle to wind a lot of times arount the z-axis during its trajectory. Indeed, in this limit we have $\Delta\varphi \to \infty$ (1 pt).

Lösung der Aufgabe 3

(a) 2 pts *Write down the Lagrangian in matrix form*
In order to write down the Lagrangian, we need the kinetic and potential energy of the two particles. The kinetic energy is

$$T = \frac{m}{2}(\dot{\xi}_1^2 + \dot{\xi}_1^2) \quad .$$

The potential energy from a spring is $\frac{k\xi_i^2}{2}$. There is one term for each of the k springs, and one term from the k_{12} spring:

$$U = \frac{k}{2}\xi_1^2 + \frac{k}{2}\xi_2^2 + \frac{k_{12}}{2}(\xi_1 - \xi_2)^2 = \frac{k + k_{12}}{2}(\xi_1^2 + \xi_2^2) - k_{12}\xi_1\xi_2 \quad .$$

The Lagrangian is thus

$$\begin{aligned}
L = T - U &= \frac{m}{2}(\dot{\xi}_1^2 + \dot{\xi}_1^2) - \frac{k + k_{12}}{2}(\xi_1^2 + \xi_2^2) + k_{12}\xi_1\xi_2. \quad \text{(1 pt)} \\
&= \frac{1}{2}\sum_{ij}\left(m_{ij}\dot{\xi}_i\dot{\xi}_j - k_{ij}\xi_i\xi_j\right)
\end{aligned}$$

where the matrices $\hat{m}$ and $\hat{k}$ have been defined as

$$\hat{m} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad , \quad \begin{pmatrix} k + k_{12} & -k_{12} \\ -k_{12} & k + k_{12} \end{pmatrix} \quad \text{(1 pt)}$$

(b) 3 pts *Determine e.o.m., write down ansatz and determine equations for the coefficients*

We have

$$\frac{\partial L}{\partial \dot{\xi}_l} = m_{li}\dot{\xi}_i \quad , \quad \frac{\partial L}{\partial \xi_l} = -k_{li}\xi_i$$

The Euler-Lagrange equations thus yield:

$$\sum_{j=1}^{2} \left(m_{ij}\ddot{\xi}_j + k_{ij}\xi_j \right) = 0 \quad \text{(1 pt)},$$

were we did some index relabeling. This equation looks like a harmonic oscillator, such that we choose the following Ansatz for the solution:

$$\vec{\xi}(t) = \vec{a}\cos(\omega t + \varphi) \quad \text{(1 pt)}.$$

Plugging this into the equations of motion, we obtain the following system of equations:

$$\sum_{j=1}^{2} \left(-\omega^2 m_{ij} + k_{ij} \right) a_j = 0.$$

If the above system of equations had full rank, the unique solution would be $\vec{a} = \vec{0}$ due to the inhomogeneous terms being zero. In order to have a nontrivial solution, the rank of the system must drop, which is equivalent to the requirement

$$\det(-\omega^2\hat{m} + \hat{k}) = \det \begin{pmatrix} k + k_{12} - m\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - m\omega^2 \end{pmatrix} \overset{!}{=} 0 \quad \text{(1 pt)}.$$

(c) 2 pts *Find the eigenfrequencies*

The above determinant condition can be solved for the frequencies ω. In order to do so, we evaluate the determinant:

$$\det \begin{pmatrix} k + k_{12} - m\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - m\omega^2 \end{pmatrix} = m^2\omega^4 - 2m\omega^2(k + k_{12}) + k^2 k_{12} \quad \text{(1 pt)}. \tag{2}$$

This is a quadratic equation in ω^2 which we can solve. We obtain

$$\omega_\pm^2 = \frac{2m(k + k_{12}) \pm 2mk_{12}}{2m^2} = \frac{k + k_{12} \pm k_{12}}{m},$$

and therefore,

$$\omega_1^2 = \frac{k}{m} \quad , \quad \omega_2^2 = \frac{k + 2k_{12}}{m} \quad \text{(1 pt)}. \tag{3}$$

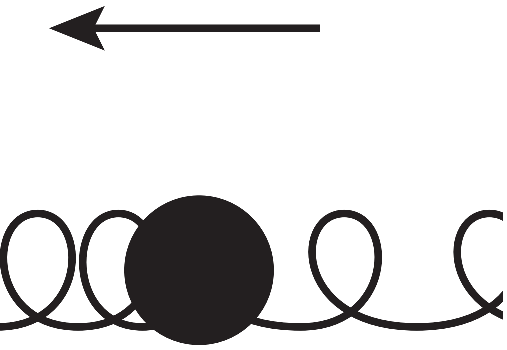
(d) [5 pts] *Find the eigenvectors*
We go back to the equation

$$\left(\begin{array}{cc} k + k_{12} - m\omega^2 & -k_{12}) \\ -k_{12} & k + k_{12} - m\omega^2 \end{array} \right) \left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \tag{4}$$

and plug in the values for ω_i^2 determined in the previous subquestion. For ω_1 this yields

$$a_1^{(1)} - a_2^{(1)} = 0$$

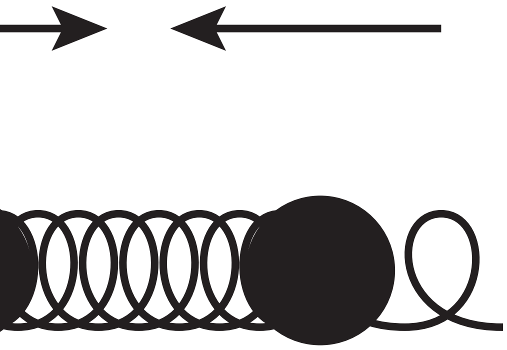
such that $a_1^{(1)} = a_2^{(1)}$ (1 pt). This represents a motion where both masses move in the same direction such that the spring in the middle does not get extended or contracted:



For ω_2 we obtain

$$a_1^{(2)} + a_2^{(2)} = 0$$

such that $a_2^{(2)} = -a_1^{(2)}$ (1 pt). This represents a motion where the masses move against each other, such that all springs get contracted and extended:



Using the normalisation condition $a_i^{(s')} m_{ij} a_j^{(s)} = \delta^{s's}$ yields the normalised vectors

$$\vec{a}^{(1)} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad , \quad \vec{a}^{(2)} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{(1 pt)}$$

(e) [3 pts] *Fix constants*
The full solution reads

$$\vec{\xi}(t) = \frac{C_1}{\sqrt{2m}} \cos(\omega_1 t + \varphi_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{C_2}{\sqrt{2m}} \cos(\omega_2 t + \varphi_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \quad \text{(1 pt)}$$

We now use the given boundary conditions to determine C_i and δ_i:

$$\left.\begin{array}{l} \dot{\xi}_1(0) = -\omega_1 \frac{C_1}{\sqrt{2m}} \sin(\varphi_1) - \omega_2 \frac{C_2}{\sqrt{2m}} \sin(\varphi_2) = 0 \\ \dot{\xi}_2(0) = -\omega_1 \frac{C_1}{\sqrt{2m}} \sin(\varphi_1) + \omega_2 \frac{C_2}{\sqrt{2m}} \sin(\varphi_2) = 0 \end{array}\right\} \Rightarrow \varphi_1 = \varphi_2 = 0 \quad \text{(1 pt)}.$$

Then,

$$\left.\begin{array}{l} \xi_1(0) = \frac{C_1}{\sqrt{2m}} + \frac{C_2}{\sqrt{2m}} = A \\ \xi_2(0) = \frac{C_1}{\sqrt{2m}} - \frac{C_2}{\sqrt{2m}} = 0 \end{array}\right\} \Rightarrow C_1 = C_2 = \frac{\sqrt{m}}{\sqrt{2}} A \quad \text{(1 pt)}.$$

The full solution with boundary conditions is thus

$$\vec{\xi}(t) = \frac{A}{2} \cos(\omega_1 t) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{A}{2} \cos(\omega_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Lösung der Aufgabe 4

(a) [2 pt] *Argue for conservation of energy and give E*
Argument: the Lagrangian does not explicitly dependent on t, so it is invariant under time translations (1 pt). As a consequence the energy is conserved.

$$E = \frac{\partial L}{\partial \dot{r}_i} \dot{r}_i - L = \frac{1}{2} m \dot{\vec{r}}^2 + \frac{\vec{a} \cdot \vec{r}}{r^3} . \quad \text{(1 pt)} \tag{6}$$

(b) [2 pt] *Show invariance action under similarity transformation*
Under the similarity transformation

$$\vec{r} \to \lambda \vec{r} , \qquad t \to \lambda^2 t . \tag{7}$$

the Lagrangian changes as $L \to \lambda^{-2} L$ and the measure as $dt \to \lambda^2 dt$ (1 pt). Thus

$$S = \int L\, dt \to \int (\lambda^{-2} L)(\lambda^2 dt) = \int L\, dt = S \tag{8}$$

the action is invariant (1 pt).

(c) 2 pt *Construct the conserved quantity with Noether's theorem*

First turn the similarity transformation into an infinitesimal transformation, by setting $\lambda = 1 + \epsilon$. Then we can identify X and Ψ_i via

$$\left.\begin{array}{lll} r_i \to (1+\epsilon) r_i = r_i + \epsilon r_i \stackrel{!}{=} r_i + \epsilon \Psi_i & \Rightarrow & \Psi_i = r_i \,, \\ t \to (1+\epsilon)^2 t \approx t + 2\epsilon t \stackrel{!}{=} t + \epsilon X & \Rightarrow & X = 2t \,. \end{array}\right\} \text{(1 pt)} \tag{9}$$

Insert this into the Noether formula (with $q_i = r_i$)

$$\begin{aligned} K &= \frac{\partial L}{\partial \dot{q}_i}(\Psi_i - X\dot{q}_i) + L\,X \\ &= \frac{\partial L}{\partial \dot{r}_i}(r_i - 2t\dot{r}_i) + L\,2t \\ &= \frac{\partial L}{\partial \dot{r}_i} r_i + \left(-\frac{\partial L}{\partial \dot{r}_i}\dot{r}_i + L\right) 2t \\ &= m r_i \dot{r}_i - 2Et \\ &= m\vec{r}\cdot\dot{\vec{r}} - 2Et \qquad \text{(1 pt)} \end{aligned} \tag{10}$$

(d) 2 pt *Find $r(t)$*

First, we show that

$$\vec{r}\cdot\dot{\vec{r}} = \frac{1}{2}\frac{d}{dt}(\vec{r}\cdot\vec{r}) = \frac{1}{2}\frac{d}{dt}(r^2) \quad \text{(1 pt)} \tag{11}$$

Alternative: Taking the inner product between $\vec{r} = r\,\hat{r}$ and

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}\,, \tag{12}$$

(the infinitesimal change $d\vec{r}$ in spherical coordinates) yields

$$\vec{r}\cdot d\vec{r} = r\,dr \qquad \Rightarrow \qquad \vec{r}\cdot\dot{\vec{r}} = r\,\dot{r} = \frac{1}{2}\frac{d}{dt}(r^2)\,. \tag{13}$$

Inserting this into the conserved quantity gives

$$K = \frac{m}{2}\frac{d}{dt}(r^2) - 2Et \quad \Rightarrow \quad \frac{d}{dt}(r^2) = \frac{2}{m}(K + 2Et) \tag{14}$$

Integrating over time gives

$$r^2 = \frac{2K}{m}t + \frac{2E}{m}t^2 + C \quad \text{(1 pt)} \tag{15}$$

so

$$r(t) = \sqrt{\frac{2E}{m}t^2 + \frac{2K}{m}t + C}\,. \tag{16}$$

(e) 3 pt *Construct Hamiltonian*

The canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i \qquad \Leftrightarrow \qquad \vec{p} = m\dot{\vec{r}}\,. \quad \text{(1 pt)} \tag{17}$$

The Hamiltonian is

$$H = \vec{p}\cdot\dot{\vec{r}} - L = \frac{\vec{p}^2}{2m} + \frac{\vec{a}\cdot\vec{r}}{r^3} \quad \text{(1 pt)} \tag{18}$$

The relation between the Hamiltonian and the energy is $H = E$ (1 pt).

(f) 1 pt *Give relation time derivative and Poisson brackets*
The relation for the function $f = f(\vec{r}, \vec{p}, t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \quad \text{(1 pt)} \tag{19}$$

NB. The definition of the Poisson bracket was given in the formula section.

(g) 2 pt *Compute Poisson bracket*
By Leibnitz' rule:

$$\{H\,,\,\vec{r}\cdot\vec{p}\} = r_i\{H\,,\,p_i\} + p_i\{H\,,\,r_i\}\,. \quad \text{(1 pt)} \tag{20}$$

The remaining two brackets are elementary:

$$\{H\,,\,r_i\} = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \tag{21}$$

$$\{H\,,\,p_i\} = -\frac{\partial H}{\partial r_i} = -\frac{a_i}{r^3} + 3\frac{\vec{a}\cdot\vec{r}}{r^5}r_i \tag{22}$$

Inserting this gives

$$\{H\,,\,\vec{r}\cdot\vec{p}\} = 2H\,. \quad \text{(1 pt)} \tag{23}$$

(h) 2 pt *Derive expression for $\vec{r}\cdot\vec{p}$*
The partial time derivative of $\vec{r}\cdot\vec{p}$ is zero, so in this case

$$\frac{d(\vec{r}\cdot\vec{p})}{dt} \underset{(f)}{=} \{H, \vec{r}\cdot p\} \underset{(g)}{=} 2H \underset{(e)}{=} 2E\,. \quad \text{(1 pt)} \tag{24}$$

Integrating once with respect to time gives

$$\vec{r}\cdot\vec{p} = 2Et + \text{const}\,. \tag{25}$$

Inserting $\vec{p} = m\dot{\vec{r}}$ (from (e)) and rearranging gives

$$m\vec{r}\cdot\dot{\vec{r}} - 2Et = \text{const}\,. \quad \text{(1 pt)} \tag{26}$$

This is the exact same conserved quantity as computed from similarity transformation in part (c).

(i) 2 pt *Determine the Hamilton equations*
The Hamilton equations are

$$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} \quad \Leftrightarrow \quad \dot{\vec{r}} = \frac{\vec{p}}{m}\,, \quad \text{(1 pt)} \tag{27}$$

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = -\frac{a_i}{r^3} + 3\frac{\vec{a}\cdot\vec{r}}{r^5}r_i \quad \Leftrightarrow \quad \dot{\vec{p}} = \frac{3(\vec{a}\cdot\vec{r})\vec{r} - r^2\vec{a}}{r^5}\,. \quad \text{(1 pt)} \tag{28}$$

(j) 2 pt *Motion along z-axis*
We have $\vec{a}$ and the initial conditions $\vec{r}(t=0)$ and $\vec{p}(t=0)$ along the z-axis. The Hamilton equations show that the change in $\vec{r}$ and $\vec{p}$ at $t=0$ are then also along the z-axis:

$$\left.\begin{array}{ll} \dot{\vec{r}}(0) = \dfrac{\vec{p}(0)}{m} & \sim \text{along } z\text{-axis} \\ \dot{\vec{p}}(0) \sim A\,\vec{r}(0) + B\,\vec{a} & \sim \text{along } z\text{-axis} \end{array}\right\} \quad \text{(1 pt)} \tag{29}$$

So after a time dt the vectors $\vec{r}$ and $\vec{p}$ are still along the z-axis, and one can repeat the argument. So $\vec{r}(t)$ and $\vec{p}(t)$ only have z-components for all time t (1 pt).

(k) $\boxed{\text{2 pt}}$ *Find $\vec{r}(t)$*

The previous question provides us with $\vec{r}(t) = (0, 0, z(t))$ for all time t. In the provided solution to (d) we can thus insert $r(t) = z(t)$, giving

$$z(t) = \sqrt{\frac{2E}{m}t^2 + \frac{2K}{m}t + C}\,. \tag{30}$$

Now we need to fix the constants.

$$\left.\begin{aligned} z(0) = \sqrt{C} \stackrel{!}{=} z_0 \\ p_z(0) = m\dot{z}(0) = \frac{K}{z_0} \stackrel{!}{=} 0 \end{aligned}\right\} \quad \begin{cases} C = z_0^2 \\ K = 0 \end{cases} \quad \color{red}{(1\,\text{pt})} \tag{31}$$

The energy at time $t = 0$ is

$$E = \frac{\vec{p}(0)^2}{2m} + \frac{\vec{a}\cdot\vec{r}(0)}{r^3(0)} = \frac{a}{z_0^2} \quad \color{red}{(1\,\text{pt})} \tag{32}$$

Inserting these constants gives the non-vanishing z-component of $\vec{r}(t)$:

$$z(t) = \sqrt{\frac{2a}{m}\frac{t^2}{z_0^2} + z_0^2}\,. \tag{33}$$