Lösung der Aufgabe 1

(a) [1 pt] *Conserved quantity for rotational symmetry*
Angular momentum $\vec{M}$. (1 pt)

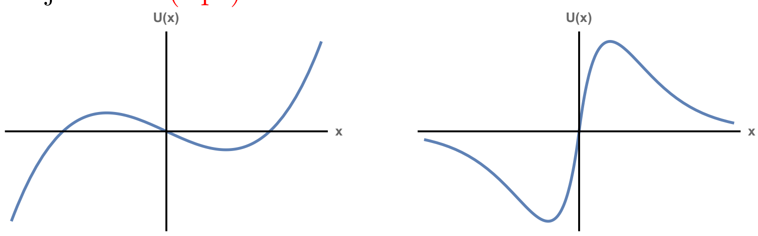
(b) [2 pt] *Two descriptions for motion on a circle*
(1 pt) Add a Lagrange multiplier $L \to L + \lambda(x^2 + y^2 - R^2)$.
(1 pt) Explicitly eliminate one coordinate. One can substitute $x = \sqrt{R^2 - y^2}$, or equivalently, change coordinates: $(x, y) \to (r, \theta)$ and then eliminate $r = R$.

(c) [1 pt] *Conserved quantities in Kepler potential*
Energy E, angular momentum $\vec{M}$, Runge-Lenz vector $\vec{A}$. (1 pt)

(d) [1 pt] *Theorem for homogeneous potentials*
The relation between average kinetic and potential energies are:

$$\langle T \rangle = \frac{n}{2} \langle U \rangle \quad \text{(1 pt)} \tag{1}$$

(e) [1 pt] *Potential for different trajectories*
The following potentials are examples which allow for both open and closed trajectories (1 pt).



(f) [2 pt] *Expression for angular momentum and energy for scattering*

$$|\vec{M}| = m v_\infty \rho \,, \quad \text{(1 pt)} \tag{2}$$

$$E = \frac{1}{2} m v_\infty^2 \quad \text{(1 pt)} \tag{3}$$

(g) [2 pt] *Effects of anharmonic corrections*
1. Frequency becomes mass dependent: FALSE (0.5 pt)
2. Frequency grows linearly with time: FALSE (0.5 pt)
3. The period becomes amplitude dependent: TRUE (0.5 pt)
4. There will be frequency multiples $2\omega, 3\omega, \ldots$: TRUE (0.5 pt)

(h) [1 pt] *Constant force oscillator solution*
The solution is the sum of homogeneous ((1 pt)) and particular ((1 pt)) solutions

$$\xi = A \cos(\omega t + \phi) + \frac{F_0}{m\omega^2} \tag{4}$$

(i) [1 pt] *Phase space volume under canonical transformation*
Phase-space volume is *invariant* under canonical transformation. (1 pt)

(j) $\boxed{\text{1 pt}}$ *Poisson bracket*

$$\{p_j, (a_k r_k)^n\} = \left(\frac{\partial p_j}{\partial p_i}\frac{\partial (a_k r_k)^n}{\partial r_i} - \frac{\partial p_j}{\partial r_i}\frac{\partial (a_k r_k)^n}{\partial p_i}\right) \tag{5}$$
$$= \delta_{ij} n (a_k r_k)^{n-1} \frac{\partial (a_\ell r_\ell)}{\partial r_i} - 0 \tag{6}$$
$$= \delta_{ij} n (a_k r_k)^{n-1} a_\ell \delta_{\ell i} \tag{7}$$
$$= n (a_k r_k)^{n-1} a_j \tag{8}$$

Hence

$$\{\vec{p}, (\vec{a}\cdot\vec{r})^n\} = n(\vec{a}\cdot\vec{r})^{n-1}\vec{a} \quad \text{(1 pt)} \tag{9}$$

(k) $\boxed{\text{1 pt}}$ *Principal moments of inertia for spherical top*
All three moments of inertia of the spherical top are identical, so the answer is one (1 pt).

Lösung der Aufgabe 2

(a) $\boxed{\text{2 pt}}$ *Lagrangian*

$$L = T - U \quad \text{with} \tag{5}$$
$$\text{(1 pt)} \quad T = \sum_{i=1}^{3} \frac{1}{2} m_i r^2 \dot{\theta}_i^2 \tag{6}$$
$$\text{(1 pt)} \quad U = \frac{1}{2} r^2 \left(k_1 u_{12}^2 + k_2 u_{23}^2 + k_3 u_{31}^2\right) \tag{7}$$
$$u_{ij}^2 = (\theta_j - \theta_i - 2\pi/3)^2 \tag{8}$$

(b) $\boxed{\text{2 pt}}$ *Matritzen*

$$L = \sum_{i=1}^{3} \frac{1}{2} m_i r^2 \dot{\xi}_i^2 - \frac{1}{2} r^2 \left(k_1(\xi_2 - \xi_1)^2 + k_2(\xi_3 - \xi_2)^2 + k_3(\xi_1 - \xi_3)^2\right) \tag{9}$$

$$\hat{m} = r^2 \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \text{ (1 pt)} \qquad \hat{k} = r^2 \begin{pmatrix} k_1 + k_3 & -k_1 & -k_3 \\ -k_1 & k_1 + k_2 & -k_2 \\ -k_3 & -k_2 & k_2 + k_3 \end{pmatrix} \text{ (1 pt)} \tag{10}$$

(c) $\boxed{\text{1 pt}}$ *Euler-Lagrange equations*
The equations are

$$m_{ij}\ddot{\xi}_j + k_{ij}\xi_j = 0 \text{ (1 pt)} \tag{11}$$

for $i = 1, 2, 3$.

(d) [2 pt] *Derive condition on Eigenfrequencies*

Solve the Euler-Lagrange equations with an Ansatz $\xi_j = a_j \cos(\omega t + \phi)$. Then

$$(-\omega^2 m_{ij} + k_{ij}) a_j \cos(\omega t + \phi) = 0 \tag{12}$$

This must hold for all times t, so

$$(-\omega^2 m_{ij} + k_{ij}) a_j = 0 \text{ (1 pt)} \tag{13}$$

We will necessarily find the trivial solution $a_j = 0$ (no oscillation), unless the matrix has less than maximal rank. This leads to the condition:

$$\det(-\omega^2 \hat{m} + \hat{k}) = 0 \text{ (1 pt)} \tag{14}$$

(e) [1 pt] *Eigenfrequenzen*

$$\omega_1^2 = 0 \qquad \omega_2^2 = \frac{3k}{m} \qquad \omega_3^2 = \frac{k}{m} + \frac{2k}{M} \text{ (1 pt)} \tag{15}$$

(f) [3 pt] *Eigenvectors* (3 pt)

$$\vec{a}^{(1)} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{a}^{(2)} = c_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{a}^{(3)} = c_3 \begin{pmatrix} -2m/M \\ 1 \\ 1 \end{pmatrix} \tag{16}$$

(g) [3 pt] *Boundary conditions*

In terms of ξ the boundary conditions are (1 pt)

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_{t=0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix}_{t=0} = \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix}. \tag{17}$$

The general solution is (1 pt)

$$\vec{\xi} = \sum_{s=1}^{3} \vec{a}^{(s)} r_s \tag{18}$$

with

$$r_1 = c_1 + d_1 t \tag{19}$$
$$r_2 = c_2 \cos(\omega_2 t + \phi_2) \tag{20}$$
$$r_3 = c_3 \cos(\omega_3 t + \phi_3) \tag{21}$$

and

$$\vec{a}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{a}^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad \vec{a}^{(3)} = \begin{pmatrix} -2m/M \\ 1 \\ 1 \end{pmatrix} \tag{22}$$

The constants are fixed to be (1 pt)

$$c_1 = 0 \tag{23}$$
$$d_1 = v_0 \frac{M}{2m + M} \tag{24}$$
$$c_2 = 0 \tag{25}$$
$$\phi_2 = \pi/2 \tag{26}$$
$$c_3 = v_0 \sqrt{\frac{m}{k}} \left(\frac{M}{2m + M} \right)^{3/2} \tag{27}$$
$$\phi_3 = \pi/2 \tag{28}$$

Lösung der Aufgabe 3

(a) 3 pt *Hamiltonian*
The canonical momentum is

$$\vec{p} = m\dot{\vec{q}} + \frac{e}{c}\vec{A} \text{ (1 pt)} \tag{10}$$

The Hamiltonian is

$$\begin{aligned} H &= \vec{p}\cdot\dot{\vec{q}} - L \text{ (1 pt)} && (11)\\ &= \left(m\dot{\vec{q}} + \frac{e}{c}\vec{A}\right)\cdot\dot{\vec{q}} - \left(\frac{1}{2}m\dot{\vec{q}}^2 + \frac{e}{c}\vec{A}(\vec{q})\cdot\dot{\vec{q}}\right) && (12)\\ &= \frac{1}{2}m\dot{\vec{q}}^2 && (13)\\ &= \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 \text{ (1 pt)} && (14)\end{aligned}$$

(b) 2 pt *Poisson bracket*
The Poisson bracket is

$$\{H\,,\,p_3\} = \sum_i \left(\frac{\partial H}{\partial p_i}\underbrace{\frac{\partial p_3}{\partial q_i}}_{0} - \frac{\partial H}{\partial q_i}\underbrace{\frac{\partial p_3}{\partial p_i}}_{\delta_{i3}}\right) = -\frac{\partial H}{\partial q_3} = 0 \text{ (1 pt)} \tag{15}$$

since H depends only on q_1 and q_2 through $A(\vec{q})$. As a result

$$\frac{dq_3}{dt} = \frac{\partial q_3}{\partial t} + \{H\,,\,p_3\} = 0 \tag{16}$$

The interpretation is that p_3 is conserved. (1 pt)

(c) 1 pt *Value of* ω
Inserting the choice of vector potential into the Hamiltonian gives

$$H = \frac{1}{2m}\left(p_1 + \frac{e}{c}\frac{B}{2}q_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{e}{c}\frac{B}{2}q_1\right)^2 + \frac{1}{2m}p_3^2 \tag{17}$$

This is equal to the given Hamiltonian for $p_3 = 0$ and

$$\omega = \frac{eB}{mc} \text{ (1 pt)} \tag{18}$$

(d) 3 pt *Canonical Transformation*
The transformation equations are

$$\left.\begin{aligned} p_1 &= \frac{\partial F}{\partial q_1} = m\omega\left(Q_1 - \tfrac{1}{2}q_2\right)\\ p_2 &= \frac{\partial F}{\partial q_2} = m\omega\left(Q_2 - \tfrac{1}{2}q_1\right)\end{aligned}\right\} \text{ (1 pt)} \tag{19}$$

$$\left.\begin{aligned} P_1 &= -\frac{\partial F}{\partial Q_1} = -m\omega\,(q_1 - Q_2)\\ P_2 &= -\frac{\partial F}{\partial Q_2} = -m\omega\,(q_2 - Q_1)\end{aligned}\right\} \text{ (1 pt)} \tag{20}$$

Solving for old coordinates and momenta gives (1 pt)

$$q_1 = Q_2 - \tfrac{1}{m\omega} P_1 \tag{21}$$
$$q_2 = Q_1 - \tfrac{1}{m\omega} P_2 \tag{22}$$
$$p_1 = \tfrac{1}{2} m\omega Q_1 + \tfrac{1}{2} P_2 \tag{23}$$
$$p_2 = \tfrac{1}{2} m\omega Q_2 + \tfrac{1}{2} P_1 \tag{24}$$

(e) [2 pt] *New Hamiltonian*
Since $\partial F/\partial t = 0$ we have $K = H$. (1 pt)
The new Hamiltonian becomes

$$K = \frac{1}{2m} P_1^2 + \frac{1}{2} m\omega^2 Q_1^2 \ \text{(1 pt)} \tag{25}$$

(f) [4 pt] *New Hamilton equations*
The *four* Hamilton equations are

$$\left.\begin{aligned} \dot{Q}_1 &= \frac{\partial H}{\partial P_1} = \frac{P_1}{m} \\ \dot{P}_1 &= -\frac{\partial H}{\partial Q_1} = -m\omega^2 Q_1 \end{aligned}\right\} \text{(1 pt)} \qquad \left.\begin{aligned} \dot{Q}_2 &= \frac{\partial H}{\partial P_2} = 0 \\ \dot{P}_2 &= -\frac{\partial H}{\partial Q_2} = 0 \end{aligned}\right\} \text{(1 pt)} \tag{26}$$

Their solutions are (1 pt)

$$Q_1 = A\cos(\omega t + \phi) \qquad Q_2 = C \tag{27}$$
$$P_1 = -m\omega A \sin(\omega t + \phi) \qquad P_2 = D \tag{28}$$

with *four* constants C, D, A, ϕ.
Using the provided expressions for q_1 and q_2 in terms of new coordinates in eq. (5)-(6) finally gives the general solutions (1 pt)

$$q_1 = C + A\sin(\omega t + \phi) \tag{29}$$
$$q_2 = A\cos(\omega t + \phi) - \tfrac{1}{m\omega} D \tag{30}$$

Lösung der Aufgabe 4

(a) [3 pt] *Symmetry*
The setup is manifestly symmetric for a translation along the boundary $y(x)$. It is given by

$$\vec{q} \to \vec{q} + \epsilon \begin{pmatrix} 1 \\ a \end{pmatrix} = \vec{q}\,' \quad \text{(1 pt)} \tag{7}$$

The kinetic term of the Lagrangian is invariant under this transformation. For the potential we look at the two quantities

$$y \to y + a\epsilon, \tag{8}$$
$$ax \to +a(x + \epsilon) = ax + a\epsilon, \tag{9}$$

such that $y < ax \Rightarrow y' < ax'$ and same for "$>$". Thus the potential is also invariant under the transformation (1 pt). The associated conserved quantity is

$$K = \frac{\partial L}{\partial \dot{x}} \cdot 1 + \frac{\partial L}{\partial \dot{y}} \cdot a = m(\dot{x} + a\dot{y}). \quad \text{(1 pt)} \tag{10}$$

This is the projection of the particle's momentum along the direction of the boundary between the two potential regions.

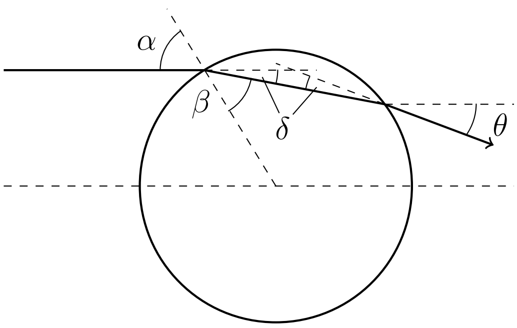
(b) 2 pt *Passing the boundary*
For $a \to 0$ the conserved quantity becomes $m\dot{x}_i = m|\vec{v}_i| \sin\varphi_i \;, \quad i \in \{1,2\}$. The conservation law can then be rephrased as $|v_1| \sin\varphi_1 = |v_2| \sin\varphi_2$ (1 pt). Energy conservation also tells us that

$$E = \frac{m}{2}|v_1|^2 + U_1 = \frac{m}{2}|v_2|^2 + U_2 \quad \Rightarrow \quad |v_2| = \sqrt{|v_1|^2 + \frac{2}{m}(U_1 - U_2)}. \tag{11}$$

Putting both conservation laws together yields

$$\frac{\sin\varphi_1}{\sin\varphi_2} = \frac{|v_2|}{|v_1|} = \sqrt{1 + \frac{2}{m|v_1|^2}(U_1 - U_2)}. \quad (1\,\text{pt}) \tag{12}$$

(c) 2 pt *Geometry of the problem*



The scattering angle θ is the angle between the outgoing particle and the $z-$axis (1 pt). From the picture we see that $\theta = 2\delta$, where δ itself is given by $\alpha - \beta$. Thus we have $\theta = 2(\alpha - \beta)$ (1 pt).

(d) 2 pt *Determine relation between ρ and θ*
We now use the formula which was proven in the first part of the exercise, and which now becomes:

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1 + \frac{2U_0}{mv_\infty^2}} = n \quad (1\,\text{pt}). \tag{13}$$

We reexpress β in terms of α and θ:

$$\sin\beta = \sin(\alpha - \frac{\theta}{2}) = \sin\alpha\cos\left(\frac{\theta}{2}\right) - \cos\alpha\sin\left(\frac{\theta}{2}\right). \tag{14}$$

putting the two previous equations together, we obtain

$$\frac{1}{n} = \cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cot\alpha. \tag{15}$$

We further express α in terms of ρ:

$$\rho^2 = a^2\sin^2\alpha \Rightarrow \cot^2\alpha = \frac{a^2 - \rho^2}{\rho^2}. \tag{16}$$

Inserting this into the square of the previous identity, we finally get after some manipulations

$$\rho^2 = a^2 \frac{n^2\sin^2\left(\frac{\theta}{2}\right)}{1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2} \quad (1\,\text{pt}). \tag{17}$$

(e) 2 pt *Determine the minimal and maximal scattering angles*
From the relation between ρ and θ we see that $\theta_{\min} = 0$, which is the case for $\rho = 0$ (1 pt). The maximal scattering angle is obtained when $\rho = a$, which implies

$$\cos\left(\frac{\theta_{\max}}{2}\right) = \frac{1}{n} \Rightarrow \theta_{\max} = 2\arccos\left(\frac{1}{n}\right) \quad \text{(1 pt)}. \tag{18}$$

(f) 2 pt *Limiting cases for $\theta_{\max}$*
The case $U_0 \to 0$ means that $n \to 1$, such that $\theta_{\max} \to 0$ (1 pt). The other case of strong potential means than $n \to \infty$, such that $\theta_{\max} \to \pi$ (1 pt) (*give half a point if the sketch misses or is wrong*).

(g) 2 pt *Determine the differential cross section*
We differentiate both sides of the identity of subquestion (d) with respect to θ and obtain

$$2\rho(\theta)\frac{\partial\rho}{\partial\theta} = \frac{n^2\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right)\right) - n^3\sin^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\left(1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right)^2} \tag{19}$$

The differential cross section is then given by

$$\begin{aligned} \mathrm{d}\sigma &= 2\pi\rho(\theta)\left|\frac{\partial\rho}{\partial\theta}\right|\mathrm{d}\theta \quad \text{(1 pt)} \\ &= \pi n^2 a^2 \frac{\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - n\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) - n\sin\left(\frac{\theta}{2}\right) + n^2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\left(1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right)^2}\mathrm{d}\theta \\ &= \pi n^2 a^2 \frac{\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right) - n\right)\left(1 - n\cos\left(\frac{\theta}{2}\right)\right)}{\left(1 - 2n\cos\left(\frac{\theta}{2}\right) + n^2\right)^2}\mathrm{d}\theta \quad \text{(1 pt)}. \end{aligned} \tag{20}$$

(*The last point can be given for either of the last two lines.*)

(h) 1 pt *Guess the total cross section*
Since the potential is spherically symmetric, the cross section of the scattering object is going to be a disc with the same radius. Its surface is πa^2 (1 pt).