Aufgabe 1 Studentische Lösung!

a) holonome Zwangsbedingung

$f(\vec{x},t)=0$

holonom-sklerenome Zwangsbedingung:

$f(\vec{x})=0$

holonom-reonome Zwangsbedingung:

$f(\vec{x},t)=0$ mit $\exists(\vec{x},t): \frac{\partial f}{\partial t}(\vec{x},t)\neq 0$

b) (i) $\mathcal{L}=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\,\dot{\phi}^2-\frac{k\exp(-\Gamma r)}{r}\right)$

$\frac{\partial \mathcal{L}}{\partial t}=0 \Rightarrow$ Energieerhaltung

Rotationssymmetrie $\Rightarrow$ Drehimpulserhaltung

(ii) $\mathcal{L}=\frac{m}{2}\dot{\vec{r}}^2-k\cos(\vec{a}\cdot\vec{r}),\quad \vec{a}=\begin{bmatrix}0\\0\\a\end{bmatrix}$

$\frac{\partial \mathcal{L}}{\partial t}=0 \Rightarrow$ Energieerhaltung

$\frac{\partial \mathcal{L}}{\partial x}=\frac{\partial \mathcal{L}}{\partial y}=0 \Rightarrow$ Impulserhaltung in x und y-Richtung (p_x, p_y)

$\Rightarrow$ Drehimpulserhaltung um die z-Achse L_z

(iii) $\frac{\partial \mathcal{L}}{\partial t}=\dot{\vec{r}}^2\frac{m}{2}-\Gamma\exp(-\Gamma t)\neq 0 \Rightarrow$ Keine Energieerhaltung

Verschiebesymmetrie und Rotationssymetrie

$\Rightarrow$ Impuls und Drehimpuls sind erhalten.

c) $0=\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}-\frac{\partial \mathcal{L}}{\partial x}=m\ddot{x}+kx$

$0=\frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}}-\frac{\partial \tilde{\mathcal{L}}}{\partial x}=\frac{d}{dt}\sqrt{m}(\sqrt{m}\dot{x}-\sqrt{k}x)+\sqrt{k}(\sqrt{m}\dot{x}-\sqrt{k}x)+2kx$

$=m\ddot{x}-\sqrt{km}\,\dot{x}+\sqrt{km}\,\dot{x}-kx+2kx$

$=m\ddot{x}+2kx \Rightarrow$ Gleichheit

$\tilde{\mathcal{L}}-\mathcal{L}=\sqrt{mk}\,\dot{x}x=\frac{d}{dt}\left(\frac{1}{2}\sqrt{mk}\,x^2\right)\Rightarrow F(t)=\frac{1}{2}\sqrt{mk}\,x^2$

d) $\hat{\Theta}=\begin{bmatrix}4a^2M & 0 & 0\\ 0 & 4a^2m & 0\\ 0 & 0 & 4a^2(m+M)\end{bmatrix}$

Da $m<M<m+M$ kann das System nicht stabil um die x-Achse mit $\Theta_{yy}<\Theta_{xx}<\Theta_{zz}$ drehen.

Um die y- und z-Achse ist eine stabile Rotation möglich.

Aufgabe 2

a) $\mathcal{L} = \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{1}{2}k(\eta_x^2+\eta_y^2) - \frac{1}{2}k\left(\sqrt{(a+\eta_x)^2+(a+\eta_y)^2}-\sqrt{2}a\right)^2$

b) Kleinwinkelnäherung:

$$\mathcal{L} = \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{1}{2}k(\eta_x^2+\eta_y^2) - \frac{1}{2}k\left(\sqrt{(a+\eta_x)^2+(a+\eta_y)^2}-\sqrt{2}a\right)^2$$

$$= \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{1}{2}k(\eta_x^2+\eta_y^2) - \frac{1}{2}k\left(\sqrt{2a^2+2a(\eta_x+\eta_y)+\underbrace{\eta_x^2+\eta_y^2}_{\approx 0}}-\sqrt{2a}\right)^2$$

$$= \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{1}{2}k(\eta_x^2+\eta_y^2) - \frac{1}{2}k\left(\cancel{\sqrt{2}a}+\frac{\sqrt{2}a}{2a}(\eta_x+\eta_y)+ - \cancel{\sqrt{2}a}\right)^2$$

$$= \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{1}{2}k(\eta_x^2+\eta_y^2) - \frac{k}{4}(\eta_x^2+2\eta_x\eta_y+\eta_y^2)$$

$$= \frac{1}{2} m(\dot{\eta}_x^2+\dot{\eta}_y^2) - \frac{3}{4}k(\eta_x^2+\eta_y^2) - \frac{1}{2}k\eta_x\eta_y$$

$$0 \stackrel{!}{=} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_x} - \frac{\partial \mathcal{L}}{\partial \eta_x} = m\ddot{\eta}_x + \frac{3}{2}k\eta_x + \frac{1}{2}k\eta_y$$

$$0 \stackrel{!}{=} \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\eta}_y} - \frac{\partial \mathcal{L}}{\partial \eta_y} = m\ddot{\eta}_y + \frac{3}{2}k\eta_y + \frac{1}{2}k\eta_x$$

$$\Rightarrow 0 = \ddot{\vec{\eta}} + \frac{k}{m}\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}\vec{\eta}$$

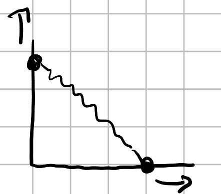
$$0 \stackrel{!}{=} \left|\begin{bmatrix} \frac{3}{2}\frac{k}{m}-\lambda & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & \frac{3}{2}\frac{k}{m}-\lambda \end{bmatrix}\right| = \left(\frac{3}{2}\frac{k}{m}-\lambda\right)^2 - \frac{1}{4}\frac{k^2}{m^2} \Rightarrow \frac{3}{2}\frac{k}{m}-\lambda = \pm\frac{1}{2}\frac{k}{m}$$

$$\Rightarrow \lambda = \frac{3}{2}\frac{k}{m} \pm \frac{1}{2}\frac{k}{m}$$

$$\Rightarrow \omega_1^2 = \frac{k}{m}$$

$$\omega_2^2 = 2\frac{k}{m}$$

d) ω_1: $\begin{bmatrix} \frac{1}{2}\frac{k}{m} & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & \frac{1}{2}\frac{k}{m} \end{bmatrix} \Rightarrow$ ev: $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$



Synchrone bewegung

ω_2: $\begin{bmatrix} -\frac{1}{2}\frac{k}{m} & \frac{1}{2}\frac{k}{m} \\ \frac{1}{2}\frac{k}{m} & -\frac{1}{2}\frac{k}{m} \end{bmatrix} =$ ev: $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Gegengleiche Bewegung

Aufgabe 3

a) $\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} = m\dot{\vec{q}} + e\vec{A} \Rightarrow \dot{\vec{q}} = \frac{\vec{p}}{m} - \frac{e}{m}\vec{A}$

$$H = \vec{p}\,\dot{\vec{q}}(\vec{p},q) - \mathcal{L} = \frac{\vec{p}^2}{m} - \vec{p}\frac{e}{m}\vec{A} - \frac{m}{2}\dot{q}(\vec{p},\vec{q})^2 - e\vec{A}\dot{\vec{q}}(\vec{p},\vec{q})$$

$$= \frac{\vec{p}^2}{m} - p\frac{e}{m}\vec{A} - \frac{m}{2}\left(\frac{\vec{p}}{m} - \frac{e}{m}\vec{A}\right)^2 - e\vec{A}\left(\frac{p}{m} - \frac{e}{m}\vec{A}\right)$$

$$= \frac{\vec{p}^2}{m} - p\frac{e}{m}\vec{A} - \frac{1}{2m}\left(\vec{p}^2 - 2e\vec{A}\vec{p} + e^2\vec{A}^2\right) - \frac{eAp}{m} + \frac{e^2\vec{A}^2}{m}$$

$$= \frac{\vec{p}^2}{2m} - \vec{p}\,\frac{e}{m}\vec{A} + \frac{e^2\vec{A}^2}{2m}$$

b) Mit $\vec{A} = \frac{B}{2}\begin{bmatrix}-q_2\\ q_1\end{bmatrix}$ folgt $H = \frac{\vec{p}^2}{2m} - \vec{p}\,\frac{e}{m}\frac{B}{2}\begin{bmatrix}-q_2\\ q_1\end{bmatrix} + \frac{e^2\begin{bmatrix}-q_2\\ q_1\end{bmatrix}^2 B^2}{2m\cdot 4}$

$$= \frac{\vec{p}^2}{2m} - \frac{e}{m}\frac{B}{2}(-p_1q_2 + p_2q_1) + \frac{e^2}{2m}\frac{B^2}{4}(q_1^2 + q_2^2)$$

$$= \frac{1}{2m}\left(p_1^2 + p_2^2 - 2\frac{eB}{2}(-p_1q_2 + p_2q_1) + e^2\frac{B^2}{4}(q_1^2 + q_2^2)\right)$$

$$= \frac{1}{2m}\left(p_1 + \frac{eB}{2}q_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{eB}{2}q_1\right)^2 \quad \text{mit } \omega = \frac{eB}{m}$$

c) $p_1 = \frac{\partial F}{\partial q_1} = m\omega\left(Q_1 - \frac{1}{2}q_2\right)$

$p_2 = \frac{\partial F}{\partial q_2} = m\omega\left(Q_2 - \frac{1}{2}q_1\right)$

$P_1 = -\frac{\partial F}{\partial Q_1} = -m\omega(q_1 - Q_2) \Rightarrow q_1 = -\frac{P_1}{m\omega} + Q_2$

$P_2 = -\frac{\partial F}{\partial Q_2} = -m\omega(q_2 - Q_1) \Rightarrow q_2 = -\frac{P_2}{m\omega} + Q_1$

$\Rightarrow p_1 = m\omega\left(Q_1 - \frac{1}{2}\left(-\frac{P_2}{m\omega} + Q_1\right)\right) = \frac{m\omega}{2}Q_1 + \frac{1}{2}P_2$

$\Rightarrow p_2 = m\omega\left(Q_2 - \frac{1}{2}\left(-\frac{P_1}{m\omega} + Q_2\right)\right) = \frac{m\omega}{2}Q_2 + \frac{1}{2}P_1$

Ja, diese Transformation ist kanonisch, da es sich um eine Lineare bijektive Abbildung von $(p, q) \to (P\ Q)$ handelt.

d) $\bar{H}(\vec{P},\vec{Q}) = H(\vec{p}(\vec{P},\vec{Q}),\vec{q}(\vec{P},\vec{Q})) + \underbrace{\frac{\partial F}{\partial t}(\vec{P},\vec{Q})}_{=0} = H(\vec{p}(\vec{P},\vec{Q}),\vec{q}(\vec{P},\vec{Q}))$

$$= \frac{1}{2m}\left(p_1 + \frac{m\omega}{2}q_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{m\omega}{2}q_1\right)^2$$

$$= \frac{1}{2m}\left(\frac{m\omega}{2}Q_1 + \frac{1}{2}P_2 + \frac{m\omega}{2}\left(-\frac{P_2}{m\omega} + Q_1\right)\right)^2 + \frac{1}{2m}\left(\frac{m\omega}{2}Q_2 + \frac{1}{2}P_1 - \frac{m\omega}{2}\left(-\frac{P_1}{m\omega} + Q_2\right)\right)^2$$

$$= \frac{\omega^2 m}{2}Q_1^2 + \frac{1}{2m}P_1^2$$

$\dot{P}_1 = -\frac{\partial \bar{H}}{\partial Q_1} = -\omega^2 m Q_1 \qquad \dot{P}_2 = -\frac{\partial \bar{H}}{\partial Q_2} = 0$

$\dot{Q}_2 = \frac{\partial \bar{H}}{\partial P_1} = \frac{1}{m}P_1 \qquad \dot{Q} = -\frac{\partial H}{\partial P_2} = 0$

$\Rightarrow \ddot{Q} = -\omega^2 Q$

$Q_1 = A\sin(\omega t) + B\cos(\omega t) \qquad Q_2 = const = D$

$P_1 = m\dot{Q} = mA\omega\cos(\omega t) - mB\omega\sin(\omega t) \qquad P_2 = const = E$

$\Rightarrow q_1 = D - A\cos(\omega t)$

$\Rightarrow q_2 = -A\sin(\omega t) - \frac{E}{m\omega}$

$\Rightarrow$ Kreisförmige Bewegung um $\begin{bmatrix} D \\ \frac{E}{m\omega} \end{bmatrix}$