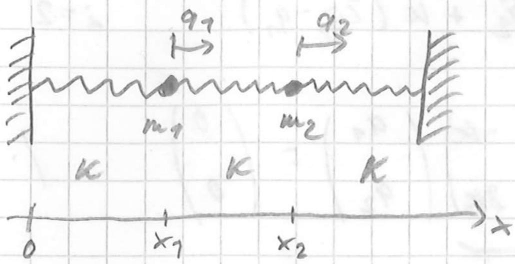
1

13)



a) Federenergie $\frac{k}{2}(x-L)^2$

mit L = entspannte Federlänge

$\Rightarrow$ pot. Energie

$$U = \frac{k}{2}(x_1-(0+L))^2 + \frac{k}{2}(x_2-(x_1+L))^2 + \frac{k}{2}(3L-(x_2+L))^2$$

$$= \frac{k}{2}\left[(x_1-L)^2+(x_2-x_1-L)^2+(x_2-2L)^2\right]$$

Umschreiben in q_1, q_2 (Auslenkungen aus Ruhelage):

$x_1 = L+q_1 \quad ; \quad \dot{x}_1 = \dot{q}_1$

$x_2 = 2L+q_2 \quad ; \quad \dot{x}_2 = \dot{q}_2$

$$\Rightarrow U = \frac{k}{2}\left[q_1^2+(q_2-q_1)^2+q_2^2\right]$$

$$T = \frac{m_1}{2}\dot{q}_1^2 + \frac{m_2}{2}\dot{q}_2^2$$

$$\Rightarrow L(q_1,q_2,\dot{q}_1,\dot{q}_2) = \frac{m_1}{2}\dot{q}_1^2+\frac{m_2}{2}\dot{q}_2^2-\frac{k}{2}\left(q_1^2+(q_2-q_1)^2+q_2^2\right)$$

L symm. in q_1, q_2: $\frac{\partial L}{\partial \dot{q}_i} = m_i\dot{q}_i \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = m_i\ddot{q}_i$

$$\frac{\partial L}{\partial q_1} = -\frac{k}{2}\left(2q_1+2(q_2-q_1)\cdot(-1)\right) = -k(2q_1-q_2)$$

$$\frac{\partial L}{\partial q_2} = -\frac{k}{2}\left(2(q_2-q_1)+2q_2\right) = -k(2q_2-q_1)$$

2/ $0 = \frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = \begin{cases} m_1 \ddot q_1 + k(2q_1 - q_2) & i=1 \\ m_2 \ddot q_2 + k(2q_2 - q_1) & i=2 \end{cases}$

b) $$\underbrace{\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}}_{=M}\begin{pmatrix} \ddot q_1 \\ \ddot q_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}}_{=K}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad | \cdot M^{-1}$$

$$M^{-1} = \begin{pmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \ddot q_1 \\ \ddot q_2 \end{pmatrix} = -M^{-1}K\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$
$$= -\begin{pmatrix} 2\frac{k}{m_1} & -\frac{k}{m_1} \\ -\frac{k}{m_2} & \frac{2k}{m_2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$
$$= -\underbrace{\begin{pmatrix} 2\omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & 2\omega_2^2 \end{pmatrix}}_{=A}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

c) Eigenwerte:
$$0 = \det(A - \lambda \mathbb{1}) = \det\begin{pmatrix} 2\omega_1^2 - \lambda & -\omega_1^2 \\ -\omega_2^2 & 2\omega_2^2 - \lambda \end{pmatrix}$$
$$= (2\omega_1^2 - \lambda)(2\omega_2^2 - \lambda) - \omega_1^2\omega_2^2$$
$$= \lambda^2 - 2\lambda(\omega_1^2 + \omega_2^2) + 3\omega_1^2\omega_2^2$$

$$\Rightarrow \lambda_\pm = (\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 + \omega_2^2)^2 - 3\omega_1^2\omega_2^2}$$
$$= (\omega_1^2 + \omega_2^2) \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + \omega_1^2\omega_2^2} \quad (\sqrt{\ } \text{ jetzt manifest positiv})$$
$$= k\left[\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \pm \sqrt{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)^2 + \frac{1}{m_1 m_2}}\right]$$

3/

Eigenvektoren:

$$\lambda_+: \begin{pmatrix} 2\omega_1^2 & -\omega_1^2 \\ -\omega_2^2 & 2\omega_2^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \lambda_+ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$2\omega_1^2 q_1 - \omega_1^2 q_2 = \left[(\omega_1^2+\omega_2^2) + \sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right] q_1$$

$$\omega_1^2 q_2 = \left[(\omega_1^2-\omega_2^2) - \sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right] q_1$$

Normierung:

$$\frac{1}{\omega_1^4} N_+^2 = 1 + \frac{(\omega_1^2-\omega_2^2)^2 + (\omega_1^2-\omega_2^2)^2 + \omega_1^2\omega_2^2 - 2(\omega_1^2-\omega_2^2)\sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}}{\omega_1^4}$$

$$= \frac{\omega_1^4 + 2(\omega_1^2-\omega_2^2)\left(\omega_1^2-\omega_2^2-\sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right) + \omega_1^2\omega_2^2}{\omega_1^4}$$

$$\vec{v}_+ = \frac{1}{N_+} \begin{pmatrix} \omega_1^2 \\ \left[(\omega_1^2-\omega_2^2) - \sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right] \end{pmatrix}$$

mit λ_+ ausgedrückt:

$$2\omega_1^2 q_1 - \omega_1^2 q_2 = \lambda_+ q_1$$

$$(2\omega_1^2 - \lambda_+) q_1 = \omega_1^2 q_2$$

Normierung: $N^2 = \omega_1^4 + (2\omega_1^2 - \lambda_+)^2$

$$\vec{v}_+ = \frac{1}{\sqrt{(2\omega_1^2-\lambda_+)^2+\omega_1^4}} \begin{pmatrix} \omega_1^2 \\ 2\omega_1^2 - \lambda_+ \end{pmatrix}$$

$$\lambda_-: \quad 2\omega_1^2 q_1 - \omega_1^2 q_2 = \lambda_- q_1$$

$$\Rightarrow \vec{v}_- = \frac{1}{\sqrt{(2\omega_1^2-\lambda_-)^2+\omega_1^4}} \begin{pmatrix} \omega_1^2 \\ 2\omega_1^2 - \lambda_- \end{pmatrix}$$

$$= \frac{1}{N_-} \begin{pmatrix} \omega_1^2 \\ \left[(\omega_1^2-\omega_2^2) + \sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right] \end{pmatrix}$$

$$\text{mit } N_-^2 = 2(\omega_1^2-\omega_2^2)\left(\omega_1^2-\omega_2^2+\sqrt{(\omega_1^2-\omega_2^2)^2+\omega_1^2\omega_2^2}\right) + \omega_1^2\omega_2^2 + \omega_1^4$$

4/ d) Rotationsmatrix $U = (\vec{v}_+, \vec{v}_-)$

mit $A_D = U^{-1} A U$

$$A_D = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

$\vec{z}$ definiert über Gleichung

$$U \cdot / \quad \ddot{\vec{z}} = -A_D \vec{z} = -U^{-1} A U \vec{z}$$

$$U \ddot{\vec{z}} = -A U \vec{z}$$

vgl. mit vorheriger Gleichung

$$\Rightarrow \quad \vec{q} = U \vec{z}$$

Bew.gl. für z: $\ddot{z}_1 = -\lambda_+ z_1$

$\ddot{z}_2 = -\lambda_- z_2$

Ansatz: $z_i = A_i \cos(\alpha_i t + \varphi_i)$

$$\ddot{z}_i = -A_i \alpha_i^2 \cos(\alpha_i t + \varphi_i) = -\alpha_i^2 z_i$$

$$\Rightarrow \quad \alpha_1 = \sqrt{\lambda_+} \quad \rightarrow z_1 = A_1 \cos(\sqrt{\lambda_+}\, t + \varphi_1)$$

$$\alpha_2 = \sqrt{\lambda_-} \quad \rightarrow z_2 = A_2 \cos(\sqrt{\lambda_-}\, t + \varphi_2)$$

e) $m_1 = m_2 = m \Rightarrow \omega_1 = \omega_2 = \omega$

$$\Rightarrow \lambda_\pm = 2\omega^2 \pm \sqrt{\omega^4} \quad \rightarrow \quad \lambda_+ = 3\omega^2$$

$$\lambda_- = \omega^2$$

$$\Rightarrow \vec{v}_+ = \frac{1}{\sqrt{2}\,\omega^2} \begin{pmatrix} \omega^2 \\ -\omega^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\vec{v}_- = \frac{1}{\sqrt{2}\,\omega^2} \begin{pmatrix} \omega^2 \\ \omega^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

5/

$$z_1 = A_1 \cos(\sqrt{3}\omega t + \varphi_1)$$
$$z_2 = A_2 \cos(\omega t + \varphi_2)$$

$$\vec{q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A_1 \cos(\sqrt{3}\omega t + \varphi_1) \\ A_2 \cos(\omega t + \varphi_2) \end{pmatrix}$$

$$q_1 = \frac{1}{\sqrt{2}} \left(A_1 \cos\left(\sqrt{\frac{3k}{m}}\, t + \varphi_1\right) + A_2 \cos\left(\sqrt{\frac{k}{m}}\, t + \varphi_2\right) \right)$$

$$q_2 = \frac{1}{\sqrt{2}} \left(-A_1 \cos\left(\sqrt{\frac{3k}{m}}\, t + \varphi_1\right) + A_2 \cos\left(\sqrt{\frac{k}{m}}\, t + \varphi_2\right) \right)$$

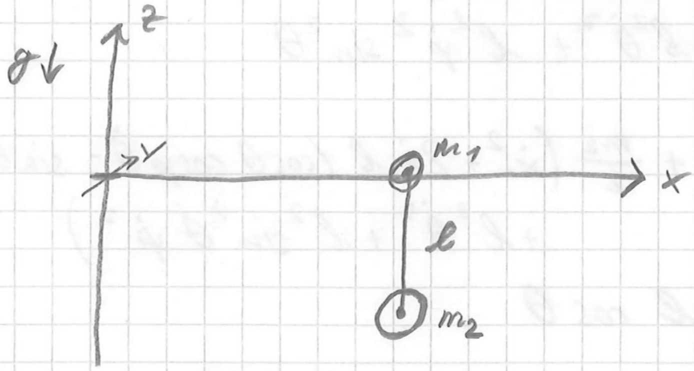
Eigenschwingungen:

2) $\sqrt{\frac{k}{m}}$: Schwingung mit gleichgerichteten Amplituden, also $x_2 - x_1 = L$ fest

1) $\sqrt{\frac{3k}{m}}$: entgegengesetzte Amplituden, also $x_1 = 3L - x_2$, symmetrisch um Mitte des Aufbaus

6/

14) Skizze:



a) Masse 1: $T_1 = \frac{m_1}{2} \dot{x}_1^2$

$U_1 = 0$

Masse 2: $T_2 = \frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2)$

$U_2 = + m_2 \cdot g \cdot z_2$

2 Massen $\Rightarrow$ 6 Freiheitsgrade

3 Zwangsbed. $\Rightarrow$ 3 freie Parameter

mögliche Wahl: $x_1, \underbrace{\theta, \varphi}$

Winkel der Auslenkung der zweiten Masse

$x_2 = x_1 + l \cdot \sin\theta \cdot \cos\varphi$

$y_2 = l \cdot \sin\theta \cdot \sin\varphi$

$z_2 = -l \cdot \cos\theta$

$\Rightarrow \dot{x}_2 = \dot{x}_1 + l \cdot \cos\theta \cos\varphi\, \dot{\theta} - l \sin\theta \sin\varphi\, \dot{\varphi}$

$\dot{y}_2 = l \cos\theta \sin\varphi\, \dot{\theta}^2 + l \sin\theta \cos\varphi\, \dot{\varphi}$

$\dot{z}_2 = + l \sin\theta\, \dot{\theta}$

$\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2 = \dot{x}_1^2 + 2\dot{x}_1 l (\cos\theta \cos\varphi\, \dot{\theta} - \sin\theta \sin\varphi\, \dot{\varphi})$

$+ l^2 \dot{\theta}^2 (\cos^2\theta \cos^2\varphi + \cos^2\theta \sin^2\varphi + \sin^2\theta)$

$+ l^2 \dot{\varphi}^2 (\sin^2\theta\, (\sin^2\varphi + \cos^2\varphi))$

$+ 2l^2 \dot{\theta}\dot{\varphi} (\cos\theta \sin\theta \cos\varphi \sin\varphi - \cos\theta \sin\theta \cos\varphi \sin\varphi)$

$$\Rightarrow \dot{x}_2^2+\dot{y}_2^2+\dot{z}_2^2 = \dot{x}_1^2 + 2\dot{x}_1 l\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right) + l^2\dot{\theta}^2 + l^2\dot{\varphi}^2\sin^2\theta$$

$$\Rightarrow L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\Big(\dot{x}_1^2 + 2\dot{x}_1 l\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right) + l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\varphi}^2\Big) + m_2 g l\cos\theta$$

b) L hängt nicht von x_1 ab.

$\Rightarrow p_1 = \frac{\partial L}{\partial \dot{x}_1}$ ist erhaltener verallgem. Impuls

$$p_1 = m_1\dot{x}_1 + m_2\left(\dot{x}_1 + l\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right)\right)$$

$$= \underbrace{(m_1+m_2)\dot{x}_1}_{\text{Schwerpunktsbewegung des gemeinsamen Systems aus } m_1 \text{ und } m_2} + m_2 l\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right)$$

c) θ: $\frac{\partial L}{\partial \dot{\theta}} = m_2\left(\dot{x}_1 l\cos\theta\cos\varphi + l^2\dot{\theta}\right)$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m_2\big(\ddot{x}_1 l\cos\theta\cos\varphi - \dot{x}_1 l\sin\theta\cos\varphi\,\dot{\theta} - \dot{x}_1 l\cos\theta\sin\varphi\,\dot{\varphi} + l^2\ddot{\theta}\big)$$

totale Zeitableitung
$\Rightarrow$ alle Größen ableiten!

heben sich weg

$$\frac{\partial L}{\partial \theta} = m_2\dot{x}_1 l\left[(-\sin\theta)\cos\varphi\,\dot{\theta} - \cos\theta\sin\varphi\,\dot{\varphi}\right] + m_2 l^2\sin\theta\cos\theta\,\dot{\varphi}^2 + m_2 g l(-\sin\theta)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m_2\big(\ddot{x}_1 l\cos\theta\cos\varphi + l^2\ddot{\theta} - l^2\sin\theta\cos\theta\,\dot{\varphi}^2 + g l\sin\theta\big) = 0$$

φ: $\quad \dfrac{\partial L}{\partial \dot\varphi} = -m_2 \dot x_1 l \sin\theta \sin\varphi + m_2 l^2 \sin^2\theta\, \dot\varphi$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} = m_2 \Big(l^2 \sin^2\theta\, \ddot\varphi + 2l^2 \sin\theta\cos\theta\, \dot\theta\dot\varphi$$
$$- \ddot x_1 l \sin\theta \sin\varphi - \dot x_1 l \cos\theta \sin\varphi\, \dot\theta$$
$$- \dot x_1 l \sin\theta \cos\varphi\, \dot\varphi \Big)$$

$$\frac{\partial L}{\partial \varphi} = m_2 \big(\dot x_1 l \cos\theta (-\sin\varphi)\, \dot\theta - \dot x_1 l \sin\theta\cos\varphi\, \dot\varphi \big)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi} - \frac{\partial L}{\partial \varphi} = m_2 \Big(l^2 \sin^2\theta\, \ddot\varphi - \ddot x_1 l \sin\theta \sin\varphi$$
$$+ 2l^2 \sin\theta\cos\theta\, \dot\theta\dot\varphi \Big) = 0$$

$$l \sin\theta\, \ddot\varphi - \ddot x_1 \sin\varphi + 2l\cos\theta\, \dot\theta\dot\varphi = 0$$

x_1: $\quad \dfrac{d}{dt} p_1 = (m_1+m_2)\ddot x_1$

$$+ m_2 l \Big(\cos\theta\cos\varphi\, \ddot\theta$$
$$- \sin\theta\cos\varphi\, \dot\theta^2$$
$$- \cos\theta\sin\varphi\, \dot\theta\dot\varphi$$
$$- \sin\theta\sin\varphi\, \ddot\varphi$$
$$- \cos\theta\sin\varphi\, \dot\theta\dot\varphi$$
$$- \sin\theta\cos\varphi\, \dot\varphi^2 \Big) = 0$$

$$(m_1+m_2)\ddot x_1 + m_2 l \Big(\cos\theta\cos\varphi\, \ddot\theta - \sin\theta\sin\varphi\, \ddot\varphi$$
$$- \sin\theta\cos\varphi\, (\dot\theta^2 + \dot\varphi^2)$$
$$- 2\cos\theta\sin\varphi\, \dot\theta\dot\varphi \Big) = 0$$

Spezialfall: $\varphi = \frac{\pi}{2}$ (m_2 schwingt entlang y-Achse)

$\cos\varphi = 0$
$\sin\varphi = 1$

$\Rightarrow$ Bew.gl.

θ: $\quad m_2 l^2 \ddot{\theta} + m_2 g l \sin\theta = 0 \Rightarrow \underbrace{\ddot{\theta} = -\frac{g}{l}\sin\theta}_{\text{Bew. gl. für Pendel}}$

x_1: $\quad p_1 = (m_1 + m_2)\dot{x}_1$

$\Rightarrow$ * System bewegt sich mit konst. Geschwindigkeit $\dot{x}_1$ entlang x-Achse

* m_2 schwingt unabhängig davon in y-Richtung

(Außerdem φ: $\quad -m_2 l \sin\theta\, \ddot{x}_1 = 0$

$\ddot{x}_1 = 0$ aus x_1-ELG

$\Rightarrow$ automatisch erfüllt)