1/

20) Euler-Lagrange-Gl:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} - \frac{\partial L}{\partial q_i} = 0$$

Betrachte nur Kraftterme

$$F_i = \frac{d}{dt}\frac{\partial U}{\partial \dot q_i} - \frac{\partial U}{\partial q_i}$$

totale Ableitung lässt sich schreiben als

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \dot q}{\partial t}\frac{\partial}{\partial \dot q} + \frac{\partial q}{\partial t}\frac{\partial}{\partial q}$$

$$= \frac{\partial}{\partial t} + \ddot q_j \frac{\partial}{\partial \dot q_j} + \dot q_j \frac{\partial}{\partial q_j}$$

also $$F_i = \frac{\partial^2 U}{\partial t\, \partial \dot q_i} + \ddot q_j \frac{\partial^2 U}{\partial \dot q_j \partial \dot q_i} + \dot q_j \frac{\partial^2 U}{\partial q_j \partial \dot q_i}$$

F darf nicht von Beschleunigung $\ddot q_i$ abhängen

$$\Rightarrow \frac{\partial^2 U}{\partial \dot q_j \partial \dot q_i} = 0$$

$\Rightarrow$ $U(q_i, \dot q_i, t)$ höchstens linear in Geschwindigkeiten $\dot q_i$

$$\Rightarrow U(q_i, \dot q_i, t) = V(q_i, t) + \dot q_i W_i(q_i, t)$$

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a) $L'(\vec{q},\dot{\vec{q}},t) = L(\vec{q},\dot{\vec{q}},t) + \frac{d}{dt} G(\vec{q},t)$

Berechne Bewegungsgleichung für L' (komp. weise)

$$\frac{\partial L'}{\partial \dot{q}_i} = \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\frac{d}{dt} G\right)$$

$$= \frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial \dot{q}_i}\left(\sum_j \frac{\partial G}{\partial q_j}\dot{q}_j + \frac{\partial G}{\partial t}\right)$$

$$= \frac{\partial L}{\partial \dot{q}_i} + \sum_j \frac{\partial G}{\partial q_j}\delta_{ij}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d}{dt}\frac{\partial G}{\partial q_i}$$

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \frac{\partial}{\partial q_i}\left(\sum_j \frac{\partial G}{\partial q_j}\dot{q}_j + \frac{\partial G}{\partial t}\right)$$

$$= \frac{\partial L}{\partial q_i} + \sum_j \dot{q}_j \frac{\partial}{\partial q_j}\frac{\partial G}{\partial q_i} + \frac{\partial}{\partial t}\frac{\partial G}{\partial q_i}$$

$$= \frac{\partial L}{\partial q_i} + \left(\sum_j \dot{q}_j \frac{\partial}{\partial q_j} + \frac{\partial}{\partial t}\right)\frac{\partial G}{\partial q_i}$$

$$= \frac{\partial L}{\partial q_i} + \frac{d}{dt}\frac{\partial G}{\partial q_i}$$

$$\Rightarrow 0 = \frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \cancel{\frac{d}{dt}\frac{\partial G}{\partial q_i}} - \frac{\partial L}{\partial q_i} - \cancel{\frac{d}{dt}\frac{\partial G}{\partial q_i}}$$

⇝ führt auf dieselbe Bewegungsgleichung

22) Sattelpunkt der Wirkung

Wirkung $S = \int_0^{t_2} dt \left(\frac{m}{2} \dot{x}^2 - \frac{D}{2} x^2 \right)$

a) Betrachte variierte Bahnkurve

$$\hat{x}(t) = x(t) + \varepsilon \eta(t)$$
$$\Rightarrow \dot{\hat{x}}(t) = \dot{x}(t) + \varepsilon \dot{\eta}(t)$$

einsetzen in S:

$$S[\hat{x}] = \int_0^{t_2} dt \left(\frac{m}{2} (\dot{x} + \varepsilon \dot{\eta})^2 - \frac{D}{2} (x + \varepsilon \eta)^2 \right)$$
$$= \int_0^{t_2} dt \left(\frac{m}{2} (\dot{x}^2 + 2\varepsilon \dot{\eta} \dot{x} + \varepsilon^2 \dot{\eta}^2) - \frac{D}{2} (x^2 + 2\varepsilon \eta x + \varepsilon^2 \eta^2) \right)$$
$$= S[x] + \varepsilon \int_0^{t_2} dt \left(m \dot{\eta} \dot{x} - D \eta x \right) + \varepsilon^2 \int_0^{t_2} dt \left(\frac{m}{2} \dot{\eta}^2 - \frac{D}{2} \eta^2 \right)$$

partielle Integration: $\int_0^{t_2} dt (\dot{\eta} \dot{x}) = [\eta \dot{x}]_0^{t_2} - \int_0^{t_2} dt\, \eta \ddot{x}$

$$\int_0^{t_2} dt\, \dot{\eta} \dot{\eta} = \underbrace{[\eta \dot{\eta}]_0^{t_2}}_{\substack{\text{beide } 0 \\ \text{wg. } \eta(0) = \eta(t_2) = 0}} - \int_0^{t_2} dt\, \eta \ddot{\eta}$$

$$\Rightarrow S[\hat{x}] = S[x] - \varepsilon \int_0^{t_2} dt\, \eta \underbrace{\left(m \ddot{x} + D x \right)}_{\text{Bew.gl. für harm. Osz.} \to 0} - \frac{\varepsilon^2}{2} \int_0^{t_2} dt\, \eta \left(m \ddot{\eta} + D \eta \right)$$

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b) Ansatz: $\eta(t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{t_2} t\right)$

$$\ddot{\eta}(t) = -\sum_{k=1}^{\infty} b_k \left(\frac{k\pi}{t_2}\right)^2 \sin\left(\frac{k\pi}{t_2} t\right)$$

$$\rightarrow m\ddot{\eta} + D\eta = \sum_{k=1}^{\infty} \left(D - m\left(\frac{k\pi}{t_2}\right)^2\right) b_k \sin\left(\frac{k\pi}{t_2} t\right)$$

$$\rightarrow \Delta S = -\frac{\varepsilon^2}{2} \sum_{k,l=1}^{\infty} \left(D - m\left(\frac{k\pi}{t_2}\right)^2\right) b_k b_l \underbrace{\int_0^{t_2} dt \, \sin\left(\frac{k\pi}{t_2} t\right) \sin\left(\frac{l\pi}{t_2} t\right)}_{= \frac{t_2}{2}\delta_{kl}}$$

$$= -\frac{\varepsilon^2}{2} \frac{t_2}{2} \sum_{k=1}^{\infty} \left(D - m\left(\frac{k\pi}{t_2}\right)^2\right) b_k^2$$

Bedingung für $\Delta S < 0$:

$$D - m\left(\frac{k\pi}{t_2}\right)^2 > 0$$

$$D > m\left(\frac{k\pi}{t_2}\right)^2$$

$$1 \leq k < \frac{t_2}{\pi}\sqrt{\frac{D}{m}}$$

$$\Rightarrow t_2 > \pi\sqrt{\frac{m}{D}} = \frac{\pi}{\omega} = \frac{T}{2}$$

vgl. Schwingungsdauer harm. Osz.: $\omega T = 2\pi \rightarrow \frac{T}{2} = \frac{\pi}{\omega}$

andererseits: betrachte $\Delta S > 0$:

$$D - m\left(\frac{k\pi}{t_2}\right)^2 < 0$$

$$k > \frac{t_2}{\pi}\sqrt{\frac{D}{m}} =: k_0$$

kann immer k finden, das diese Ungleichung erfüllt

$\rightarrow$ Für $\eta_1(t) = \sum_{k=1}^{\lfloor k_0 \rfloor} b_k \sin\left(\frac{k\pi}{t_2} t\right)$: $\Delta S < 0$

$\eta_2(t) = \sum_{k=\lceil k_0 \rceil}^{\infty} b_k \sin\left(\frac{k\pi}{t_2} t\right)$: $\Delta S > 0$

$\Rightarrow S$ weder minimal noch maximal

(Für $t_2 < \frac{T}{2}$: $k_0 < 1$: $\Delta S \geq 0$, $S[x]$ minimal)

Theo B Blatt 9

23) Ring-Oszillator

a) Massenpunkt auf Kreisbahn

$$x = R\cos\varphi \;;\quad \dot{x} = -R\sin\varphi\,\dot{\varphi}$$
$$y = R\sin\varphi \;;\quad \dot{y} = R\cos\varphi\,\dot{\varphi}$$

$$T_m = \frac{m}{2}(\dot{x}^2+\dot{y}^2) = \frac{m}{2}R^2\dot{\varphi}^2$$

also $T = \frac{m}{2}R^2(\dot{\varphi}_1^2+\dot{\varphi}_2^2+\dot{\varphi}_3^2)$

pot. Energie

In Ruhelage Feder entspannt

$$\Rightarrow U_{12} = \frac{k}{2}\Big(\underbrace{R\psi_2}_{\text{zurückgelegten Weg auf Kreisbahn}} - R\psi_1 - \overset{\text{Federlänge}}{L}\Big)^2$$

Absolutwinkel ψ

Absolutwinkel in Ruhelage um $\frac{2\pi}{3}$ verschoben

$$\psi_{2,1} = \psi_1 + \frac{2\pi}{3}$$

kreisumfang

Federlänge $3\cdot L = 2\pi R$

$$L = \frac{2\pi}{3}R$$

in Relativwinkel φ_2

oBdA $\psi_1 = \varphi_1 \Rightarrow \psi_2 = \varphi_2 + \frac{2\pi}{3}$

also $U_{12} = \frac{k}{2}R^2(\varphi_2-\varphi_1)^2$

analog U_{23}, U_{31}

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$$\Rightarrow L = \frac{m}{2}R^2(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2) - \frac{\kappa}{2}R^2\left((\varphi_2-\varphi_1)^2 + (\varphi_3-\varphi_2)^2 + (\varphi_1-\varphi_3)^2\right)$$

$$= \frac{m}{2}R^2(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2) - \kappa R^2\left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - \varphi_1\varphi_2 - \varphi_2\varphi_3 - \varphi_3\varphi_1\right)$$

b) Bew. gl.:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_i} = mR^2\ddot\varphi_i$$

$$\frac{\partial L}{\partial \varphi_1} = -\kappa R^2(2\varphi_1 - \varphi_2 - \varphi_3)$$

$$\frac{\partial L}{\partial \varphi_2} = -\kappa R^2(2\varphi_2 - \varphi_1 - \varphi_3)$$

$$\frac{\partial L}{\partial \varphi_3} = -\kappa R^2(2\varphi_3 - \varphi_1 - \varphi_2)$$

$$\ddot{\vec\varphi} = -\frac{\kappa}{m}\underbrace{\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}}_{A}\vec\varphi \overset{!}{=} -\omega^2\vec\varphi$$

mit Ansatz f. $\vec\varphi$:
$\vec\varphi = \vec B \sin(\omega t + \varphi_0)$

EW: $\det(A - \lambda \mathbb{1})$

$$= (2-\lambda)^3 - 1 - 1 - (2-\lambda)\cdot 3$$

$$= -\lambda^3 + 6\lambda^2 - 12\lambda + \cancel{8} - \cancel{8} + 3\lambda = 0$$

$$-\lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\lambda(\lambda-3)^2 = 0$$

$$\Rightarrow \lambda_1 = 0$$

$$\lambda_{2,3} = 3$$

$\Rightarrow$ Eigenfrequenzen:

$$\omega_1 = 0$$

$$\omega_{2,3} = \sqrt{\frac{3\kappa}{m}}$$

7/ Eigenvektoren:

zu $\lambda_1 = 0$: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \vec{0}$

$$2\varphi_1 - \varphi_2 - \varphi_3 = 0 \quad \Rightarrow \varphi_3 = 2\varphi_1 - \varphi_2$$

$$-\varphi_1 + 2\varphi_2 - 2\varphi_1 + \varphi_2 = 0$$

$$\varphi_2 = \varphi_1 \quad \Rightarrow \varphi_3 = \varphi_1$$

$$\Rightarrow \vec{v}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

zu $\lambda_2 = 3$: $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 3 \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$

$$\Rightarrow \varphi_1 + \varphi_2 + \varphi_3 = 0 \quad \text{(alle 3 Gl.)}$$

Versuche einen EV zu finden:

$$\varphi_3 = 0 \quad \Rightarrow \varphi_1 = -\varphi_2$$

$$\Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}$$

$\vec{v}_3$ aus Orthogonalität:

$$\vec{v}_2 \cdot \vec{v}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = 0$$

$$\varphi_1 - \varphi_2 = 0 \Rightarrow \varphi_1 = \varphi_2$$

$$\vec{v}_1 \cdot \vec{v}_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_1 \\ \varphi_3 \end{pmatrix} = 0$$

$$2\varphi_1 + \varphi_3 = 0 \quad \Rightarrow \varphi_3 = -2\varphi_1$$

$$\Rightarrow \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}$$

($\vec{v}_2, \vec{v}_3$ nicht eindeutig bestimmt)

$\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ auch möglich + alles zyklisch vertauscht

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c) allg. Lösung:

$\omega_1 = 0 \Rightarrow$ Schwingungsansatz funktioniert nicht

Bew.gl. für $\varphi_1 = \varphi_2 = \varphi_3$ $(\Rightarrow U = 0)$:

$$mR^2 \ddot{\varphi}_1 = 0$$

$$\Rightarrow \varphi = C_1 t + C_0$$

also zusammen

$$\vec{\varphi}(t) = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\vec{v}_1} \cdot (C_1 t + C_0)$$

$$+ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{\vec{v}_2} \cdot B_2 \sin(\omega_2 t + \varphi_{02})$$

$$+ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}_{\vec{v}_3} \cdot B_3 \sin(\omega_3 t + \varphi_{03})$$

Normierung nicht wichtig, kann in Redefinition der Konstanten absorbiert werden

Sattelpunkt der Wirkung

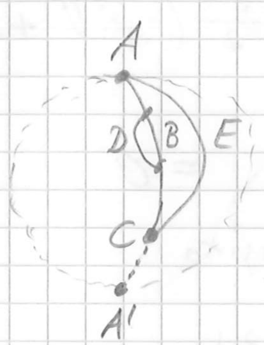
Hamilton-Wirkung $S = \int_P^R L(x, \dot{x}, t)\, dt$

mit $P(x_P, t_P)$, $R(x_R, t_R)$ festgehalten

$\rightarrow$ Gesamtzeit $t_R - t_P$ fest

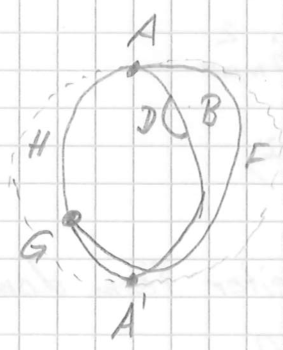
"Kinetischer Fokus"

Bsp: Trajektorie (Großkreis-Segment) auf Kugel



Strecke kleiner als halber Großkreis (A' liegt gegenüber von A)

ABC immer kürzeste Strecke, unabh. ob lokale (ADC) oder globale Variation (AEC)



Strecke größer als halber Großkreis

ABG kürzeste Strecke unter lokalen Variationen (ADG) aber global "andersherum" (AHG) kürzer, also auch AFG kürzer.

$\rightarrow$ Fokuspunkt A' (Übergang Minimum $\rightarrow$ Sattelpunkt)

harm. Osz: Startpunkt bei $(t=0, x=0, \dot{x}>0)$

(1.) Fokuspunkt bei $(t=\frac{T}{2}, x=0, \dot{x}<0)$

Betrachte Pfad-Variationen

$x(t) = x_0(t) + \varepsilon\varphi(t)$ mit $x_0(t)$ tatsächliche Bahnkurve

und $\varphi(t)$ mit $\varphi(t_P) = \varphi(t_R) = 0$, ansonsten beliebig

entwickle L als Funktion von ε, $L = \frac{m}{2}\dot{x}^2 - U(x,t)$

$$L = \underbrace{L_0}_{=L|_{\varepsilon=0}} + \varepsilon \frac{\partial L}{\partial \varepsilon}\Big|_{\varepsilon=0} + \frac{\varepsilon^2}{2}\frac{\partial^2 L}{\partial \varepsilon^2}\Big|_{\varepsilon=0} + \dots$$

$$= L_0 + \varepsilon(-\varphi U' + m\dot{x}_0\dot{\varphi}) + \frac{\varepsilon^2}{2}(-\varphi^2 U'' + m\dot{\varphi}^2)$$

$$\delta S = \varepsilon \int_P^R dt\,(\underbrace{-\varphi U'}_{\underset{ELG}{=} +\varphi m\ddot{x}_0 \xrightarrow[\text{Int.}]{\text{part}} -m\dot{\varphi}\dot{x}_0} + m\dot{x}_0\dot{\varphi}) = 0$$

$$\delta^2 S = \frac{\varepsilon^2}{2}\int_P^R dt\,(-\varphi^2 U'' + m\dot{\varphi}^2)$$

$$\varphi(t) = \int_P^t d\tilde{t}\,\dot{\varphi}(\tilde{t}) \leq (t - t_P)\,\dot{\varphi}_{max} \quad \text{da } \varphi(t_P) = 0$$

$$\Rightarrow \left|\int_P^R dt\,(-\varphi^2 U'')\right| < (t - t_P)^3\,\dot{\varphi}_{max}^2$$

$$\int_P^R dt\, m\dot{\varphi}^2 = m(t - t_P)\langle\dot{\varphi}^2\rangle$$

falls Bereich genügend klein, zweiter Term dominant

$$\Rightarrow \delta^2 S \to \frac{\varepsilon^2}{2} m \int_P^R \dot{\varphi}^2\,dt > 0$$

$\to$ Wirkung minimal für kleine Zeitstücke

aus $L = 2T - E$ f. konservatives Potential $U = U(x)$

" $\to \delta^2 S = \frac{\varepsilon^2}{2}\cdot 2m\int_P^R \dot{\varphi}^2\,dt > 0$ "

$\Rightarrow$ Sattelpunkt verknüpft mit Verletzung der Energieerhaltung (von variierten Pfaden) $\leftrightarrow$ $(t_P - t_R)$ festgehalten, E nicht in Hamilton-Prinzip the nature of the stationary value of the action are also useful for other purposes in classical and semiclassical mechanics,[28] but are not discussed in this paper.

Appendix A adapts the results to the important Maupertuis action W. Appendix B gives examples of both Hamilton and Maupertuis action for two-dimensional motion. Appendix C discusses open questions on the stationary nature of action for some newer action principles.

II. KINETIC FOCUS

This section introduces the concept of *kinetic focus* due to Jacobi,[22] which plays a central role in determining the nature of the stationary action. We start with an analogous example taken verbatim from Whittaker,[36] an analysis of the relative length along different paths. Whittaker employs the Maupertuis action principle (discussed in our Appendix A), which requires fixed total energy along trial paths, not fixed travel time as with the Hamilton action principle. In force-free systems the value of the Maupertuis action is proportional to the path length. The term kinetic focus is defined formally later in this section. Figures 1(a) and 1(b) illustrate this example. Whittaker wrote that "A simple example illustrative of the results obtained in this article is furnished by the motion of a particle confined to a smooth sphere under no forces. The trajectories are great-circles on the sphere and the [Maupertuis] action taken along any path (whether actual or trial) is proportional to the length of the path. The kinetic focus of any point A is the diametrically opposite point A' on the sphere, because any two great circles through A intersect again (for the first time) at A'. The theorems of this article amount, therefore in this case to the statement that an arc of a great circle joining any two points A and B on the sphere is the shortest distance from A to B when (and only when) the point A' diametrically opposite to A does not lie on the arc, that is, when the arc in question is less than half a great-circle."

The elaboration of this analogy is discussed in the captions of Figs. 1(a) and 1(b) using equilibrium lengths of a rubber band on a slippery spherical surface.

For a contrasting example, we apply a similar analysis to free-particle motion in a flat plane. In this case the length of the straight path connecting two points is a minimum no matter how far apart the endpoints. A rubber band stretched between the endpoints on a slippery surface will always snap back when deflected in any manner and released. An alternative straight path that deviates slightly in direction at the initial point A continues to diverge and does not cross the original path again. Therefore no kinetic focus of the initial point A exists for the original path.

Note that on both the sphere and the flat plane there is no path of true maximum length between any two separated points. The length of any path can be increased by adding wiggles.

How do we find the kinetic focus? In Fig. 1(a) we place the terminal point C at different points along a great circle path between A and A'. When C lies between A and A', every nearby alternative path such as AEC is not a true path (a path of minimum length), because it does not lie along a great circle. When terminal point C reaches A', there is suddenly more than one alternative great circle path connecting A and A'. (In this special case an infinite number of alternative great circle paths connect A and A'.) Any alternative great circle path between A and A' can be moved sideways to coalesce with the original path ABA'. The kinetic focus is defined by the existence of this coalescing alternative true path. As the final point C moves away from the initial point A, the kinetic focus A' is defined as the earliest terminal point at which two true paths can coalesce.

The term kinetic focus in mechanics derives from an analogy[37] to the focus in optics, that point A' at which rays

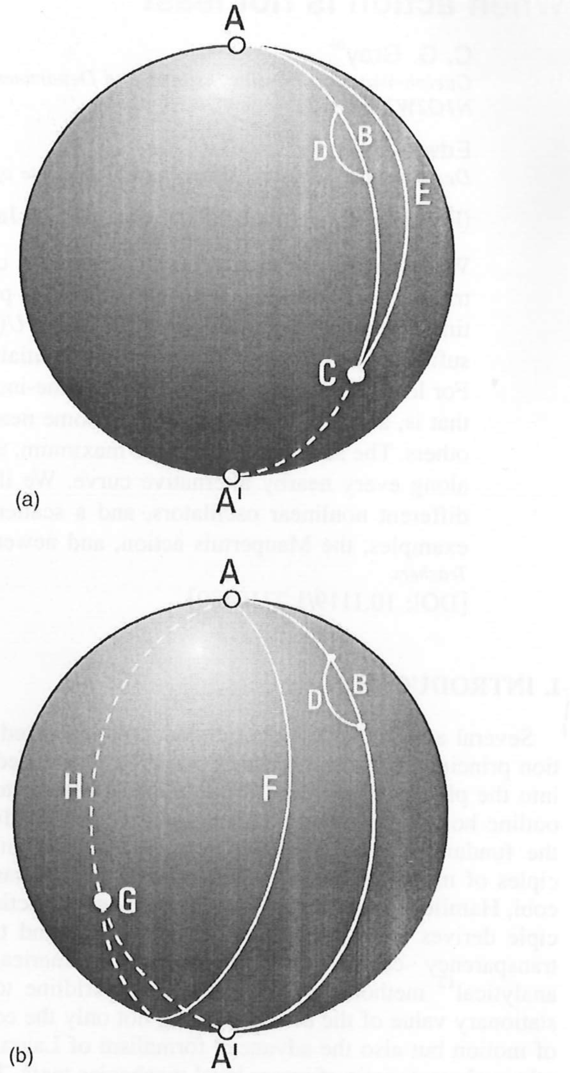


Fig. 1. (a) On a sphere the great circle line ABC starting from the north pole at A is the shortest distance between two points as long as it does not reach the south pole at A'. On a slippery sphere a rubber band stretched between A and C will snap back if displaced either locally, as at D, or by pulling the entire line aside, as along AEC. The point A' is called the *antipode* of A or in general the *kinetic focus* of A. We say that if a great circle path terminates before the kinetic focus of its initial point, the length of the great circle path is a minimum. (b) If the great circle $ABA'G$ passes through antipode A' of the initial point A, then the resulting line has a minimum length only when compared with some alternative lines. For example on a slippery sphere a rubber band stretched along this path will still snap back from local distortion, as at D. However, if the entire rubber band is pulled to one side, as along AFG, then it will not snap back, but rather slide over to the portion AHG of a great circle down the backside of the sphere. With respect to paths like AFG, the length of the great circle line $ABA'G$ is a maximum. With respect to all possible variations we say that the length of path $ABA'G$ is a saddle point. If a great circle path terminates beyond the kinetic focus of its initial point, the length of the great circle path is a saddle point.