Quiz 2 – Block 2

Geodäte auf Hyperboloid

Parametrisierung der Oberfläche

$$x = \cosh u \, \cos \varphi$$
$$y = \cosh u \, \sin \varphi$$
$$z = \sinh u$$

a) Infinitesimale Elemente

$$dx = \sinh u \cos \varphi \, du - \cosh u \sin \varphi \, d\varphi$$
$$dy = \sinh u \sin \varphi \, du + \cosh u \cos \varphi \, d\varphi$$
$$dz = \cosh u \, du$$

b) Wegelement als

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

$$= \Big[\sinh^2 u \cos^2 \varphi (du)^2 + \cosh^2 u \sin^2 \varphi (d\varphi)^2 - 2 \sinh u \cos \varphi \cosh u \sin \varphi \, du \, d\varphi + \sinh^2 u \sin^2 \varphi (du)^2 + \cosh^2 u \cos^2 \varphi (d\varphi)^2 + 2 \sinh u \sin \varphi \cosh u \cos \varphi \, du \, d\varphi + \cosh^2 u (du)^2 \Big]^{1/2}$$

$$= \left[(\sinh^2 u + \cosh^2 u)(du)^2 + \cosh^2 u (d\varphi)^2 \right]^{1/2}$$

Länge $L = \int_1^2 ds = \int_1^2 \sqrt{(\sinh^2 u + \cosh^2 u)(du)^2 + \cosh^2 u (d\varphi)^2}$

$$= \int_1^2 du \sqrt{(\sinh^2 u + \cosh^2 u) + \cosh^2 u \left(\frac{d\varphi}{du}\right)^2}$$

c) $\Rightarrow \quad F = \sqrt{(\sinh^2 u + \cosh^2 u) + \cosh^2 u \left(\frac{d\varphi}{du}\right)^2}$ ✓

Euler-Lagrange-Gleichung:

$$\frac{d}{du}\frac{\partial F}{\partial \varphi'} - \frac{\partial F}{\partial \varphi} = 0$$ ✓

F hängt nicht von φ ab: $\quad \frac{\partial F}{\partial \varphi} = 0$ ✓

$$\Rightarrow \quad \frac{\partial F}{\partial \varphi'} = \text{const.}$$ ✓

$$\frac{\cosh^2 u \cdot \cancel{2} \cdot \varphi'}{\cancel{2}\sqrt{(\sinh^2 u + \cosh^2 u) + \cosh^2 u \, \varphi'^2}} = c$$ ✓

$$\varphi'^2 \cdot \cosh^4 u = c^2 (\sinh^2 u + \cosh^2 u + \cosh^2 u \, \varphi'^2)$$

$$\varphi'^2 \cosh^2 u (\cosh^2 u - c^2) = c^2 (\sinh^2 u + \cosh^2 u)$$

$$\frac{d\varphi}{du} = \varphi' = \pm \frac{c\sqrt{\sinh^2 u + \cosh^2 u}}{\cosh u} \sqrt{\frac{1}{\cosh^2 u - c^2}}$$ ✓✓✓

d) $\int_{\varphi_s}^{\varphi} d\tilde{\varphi}$

$$= \varphi - \varphi_s = \pm \int_{u_s}^{u} d\tilde{u} \, \frac{c\sqrt{\sinh^2 u + \cosh^2 u}}{\cosh u} \sqrt{\frac{1}{\cosh^2 u - c^2}}$$ ✓✓