Quiz 2 - Block 3

Geodäte auf Kugel

Parametrisierung der Oberfläche
(hier die in der Geografie verwendete zu Längen- und Breitengrad)

$$x = r \cos\theta \cos\varphi$$
$$y = r \cos\theta \sin\varphi$$
$$z = r \sin\theta$$

a) Infinitesimale Elemente:

$$dx = -r \sin\theta \cos\varphi \, d\theta - r \cos\theta \sin\varphi \, d\varphi \quad \checkmark$$
$$dy = -r \sin\theta \sin\varphi \, d\theta + r \cos\theta \cos\varphi \, d\varphi \quad \checkmark$$
$$dz = r \cos\theta \, d\theta \quad \checkmark$$

b) Wegelement ds

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \quad \checkmark$$

$$= \Big[r^2 \sin^2\theta \cos^2\varphi (d\theta)^2 + r^2 \cos^2\theta \sin^2\varphi (d\varphi)^2 + 2r^2 \sin\theta \cos\varphi \cos\theta \sin\varphi \, d\theta \, d\varphi + r^2 \sin^2\theta \sin^2\varphi (d\theta)^2 + r^2 \cos^2\theta \cos^2\varphi (d\varphi)^2 - 2r^2 \sin\theta \sin\varphi \cos\theta \cos\varphi \, d\theta \, d\varphi + r^2 \cos^2\theta (d\theta)^2 \Big]^{1/2} \quad \checkmark\checkmark$$

$$= \left[r^2 (d\theta)^2 + r^2 \cos^2\theta (d\varphi)^2 \right]^{1/2} \quad \checkmark$$

Länge $L = \int_{\gamma}^{2} ds \overset{\checkmark}{=} \int_{\gamma}^{2} r \sqrt{(d\theta)^2 + \cos^2\theta \, (d\varphi)^2}$

$$= \int_{\gamma}^{2} r \, d\theta \sqrt{1 + \cos^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} \quad \checkmark$$

c) $\Rightarrow F = \sqrt{r^2 + r^2 \cos^2\theta \, \varphi'^2}$ ✓

Euler-Lagrange-Gleichungen: $\frac{d}{d\theta}\frac{\partial F}{\partial \varphi'} - \frac{\partial F}{\partial \varphi} = 0$ ✓

F hängt nicht von φ ab: $\frac{\partial F}{\partial \varphi} = 0$ ✓

$\Rightarrow \frac{\partial F}{\partial \varphi'} = \text{const.}$ ✓

$$\frac{2 r^2 \cos^2\theta \, \varphi'}{2\sqrt{r^2 + r^2 \cos^2\theta \, \varphi'^2}} = c$$ ✓

$$r^4 \cos^4\theta \, \varphi'^2 = c^2 r^2 (1 + \cos^2\theta \, \varphi'^2)$$

$$\varphi'^2 (r^4 \cos^4\theta - c^2 r^2 \cos^2\theta) = c^2 r^2$$

$$\frac{d\varphi}{d\theta} = \varphi' = \pm c r \left(r^4 \cos^4\theta - c^2 r^2 \cos^2\theta\right)^{-1/2}$$ ✓✓

d) $\int_{\varphi_s}^{\varphi} d\tilde{\varphi}$

$$= \varphi - \varphi_s = \pm \int_{\theta_s}^{\theta} d\tilde{\theta} \, c r \left(r^4 \cos^4\tilde{\theta} - c^2 r^2 \cos^2\tilde{\theta}\right)^{-1/2}$$ ✓✓