Theo B, ÜB 1

A1)

a) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \varepsilon_{ijk}\,\varepsilon_{lmk}\, a_i b_j c_l d_m$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_i b_j c_l d_m = a_i c_i b_j d_j - a_i d_i b_j c_j$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

b) $[\vec{\nabla}\times(\vec{r}\times\vec{a})]_k = \varepsilon_{ijk}\,\partial_i (\vec{r}\times\vec{a})_j$

$$= \underbrace{\varepsilon_{ijk}}_{=-\varepsilon_{ikj}}\, \varepsilon_{lmj}\, \underbrace{\partial_i r_l}_{=\delta_{il}}\, a_m = -\varepsilon_{ikj}\,\varepsilon_{imj}\, a_m$$

$$= -(\underbrace{\delta_{ii}}_{=3}\delta_{km} - \underbrace{\delta_{im}\delta_{ik}}_{=\delta_{km}})\, a_m = -2a_k$$

$$= -2(\vec{a})_k$$

c) $\vec{\nabla}\cdot(\vec{a}\times(\vec{r}\times\vec{b})) = \partial_k(\varepsilon_{ijk}\, a_i\, \varepsilon_{lmj}\, r_l\, b_m)$

$$= -\varepsilon_{ikj}\,\varepsilon_{lmj}\,\delta_{kl}\, a_i b_m = -(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl})\,\delta_{kl}\, a_i b_m$$

$$= -(\underbrace{\delta_{ik}\delta_{km}}_{=\delta_{im}} - \delta_{im}\cdot 3)\, a_i b_m = 2a_i b_i = 2\vec{a}\cdot\vec{b}$$

A2

(1): $y'(x) = \alpha(x) y(x)$, $A'(x) = \alpha(x)$

a) zeige, dass $y(x) = C \cdot \exp(A(x))$ (1) löst:

$$y'(x) = C \cdot A'(x) \exp(A(x)) = \alpha(x)\, C \exp(A(x)) = \underline{\underline{\alpha(x)\, y(x)}}$$

oder: $(1) \Rightarrow \frac{y'}{y} = \alpha(x) \Rightarrow \int \frac{dy}{y} = \int \alpha(x)\, dx$

$$\Rightarrow \ln(y(x)) - \ln C = A(x)$$

$$\Rightarrow y(x) = \exp(A(x) + \ln C) = C \exp(A(x))$$

Anfangsbed.: $y(x_0) = C \exp(A(x_0)) = y_0$

b) $x_0 = 1$, $y_0 = 2$

$x y' + y = 0 \Leftrightarrow y' = -\frac{1}{x} y$ $(x \neq 0)$

$\Rightarrow \alpha(x) = -\frac{1}{x} \Rightarrow A(x) = -\ln x + \ln C_b$

wähle $C_b = 0$, da $A(x)$ beliebige Stammfkt. ist

$\Rightarrow y(x) = C \exp(-\ln x) = \frac{C}{x}$

$y(1) \overset{!}{=} 2 = \frac{C}{1} \Rightarrow C = 2$

$\Rightarrow y(x) = \frac{2}{x}$

check: $-x \cdot \frac{2}{x^2} + \frac{2}{x} = 0$ ✓

A2 F)

c) $y' = -\cos x \, y \quad , \; y_0 = 1 \; , \; x_0 = 0$

$\Rightarrow A(x) = -\int dx \cos x \cong -\sin x$

$\Rightarrow y(x) = C \exp(-\sin x) \quad , \; y(0) \overset{!}{=} 1 = C \exp(0)$

$\Rightarrow C = 1$

$\Rightarrow y(x) = \exp(-\sin x)$

d) $y' = -\frac{1-x^2}{x} y \quad , \; y_0 = 1 = x_0$

$\Rightarrow A(x) = -\int \frac{1-x^2}{x} dx = \int (x - \frac{1}{x}) dx = \frac{x^2}{2} - \ln x$

$\Rightarrow y(x) = C \exp(\frac{x^2}{2} - \ln x) = \frac{C}{x} \exp(\frac{x^2}{2})$

$y(1) \overset{!}{=} 1 = C \exp(\frac{1}{2}) = C\sqrt{e} \quad \Rightarrow C = \frac{1}{\sqrt{e}}$

$\Rightarrow y(x) = \frac{1}{x\sqrt{e}} \exp(\frac{x^2}{2})$

check: $y' = \frac{1}{\sqrt{e}} \left(-\frac{1}{x^2} \exp(\frac{x^2}{2}) + \frac{1}{x} x \exp(\frac{x^2}{2}) \right)$

$= (-\frac{1}{x} + x) y(x) = -\frac{1-x^2}{x} y(x)$

A3

a) Betrachte Impulsänderung von Rakete + Sprit im Intervall $[t, t+dt]$.

Zeit t: $p(t) = m(t)\,v(t) = mv$

$t+dt$: $p_{Rakete} = p_R = (m+dm)(v+dv)$

$p_{Sprit} = p_S = -dm\,(v - v_R)$

$$\Rightarrow p(t+dt) = p_R + p_S \simeq p(t) + dp$$

$$= [mv + m\,dv + v\,dm] - v\,dm + v_R\,dm$$

$$\Rightarrow dp = m\,dv + v_R\,dm$$

Gravitationskraft wirkt auf gesamte Masse m:

$$F = -mg$$

Kraft entspricht Impulsänderung:

$$\frac{dp}{dt} = m\frac{dv}{dt} + v_R\frac{dm}{dt} = -mg$$

$$\Rightarrow m\frac{dv}{dt} = -v_R\frac{dm}{dt} - mg \qquad (1)$$

b) Masseverlust $\propto t$

$$\Rightarrow m(t) = m(0)\,(1 - f\,t), \quad f: \frac{\text{Masseverlust}}{\text{Sekunde}}$$

$$\Rightarrow \frac{dm}{dt} = -m_0 f$$

$$(1) \Rightarrow m_0(1-ft)\frac{dv}{dt} = +v_R m_0 f - m_0(t - ft)g$$

$$\Rightarrow \frac{dv}{dt} = \frac{v_R f}{1-ft} - g$$

$$\Rightarrow v(t) - \underbrace{v(0)}_{=0} = \int_0^t dt' \left[\frac{v_R f}{1-ft'} - g\right]$$

A3 F

b F) $v(t) = \frac{v_R}{f} f \int_0^{ft} d(ft') \frac{1}{1-ft'} - gt$

$= -v_R \left[\ln(1-ft') \right]_0^t - gt$

$= -v_R \ln(1-ft) - gt$

c) Suche eine Lösung für $v(t_F) = v_F = 11\,300 \frac{m}{s}$

aus (b): $\frac{m(t_F)}{m_0} = 1 - ft_F$

$\Rightarrow v_F = -v_R \ln \frac{m_F}{m_0} - gt_F \qquad (2)$

Beachte, dass nach $t_{End} = \frac{1}{f} = 60\,s$ gesammte Masse verbraucht ist. Nehme an, dass

$gt_F \approx gt_{End}$

$\Rightarrow (2): \ln \frac{m_F}{m_0} \approx -\frac{v_F + g\, t_{End}}{v_R}$

$\Rightarrow \frac{m_0}{m_F} \approx \exp\left(\frac{v_F + g/f}{v_R}\right) \approx 312$

"exakte" Lösung: aus (b): $t_F = \frac{1}{f}\left(1 - \frac{m_F}{m_0}\right)$

$\Rightarrow (2): v_F = -v_R \ln \frac{m_F}{m_0} - \frac{g}{f}\left(1 - \frac{m_F}{m_0}\right)$

$\Rightarrow \frac{m_F}{m_0} = -\frac{1}{\xi} W\left[-\frac{\xi}{\exp\left(\xi + \frac{v_F}{v_R}\right)}\right] = 311.8 \quad , \xi = \frac{g}{f v_R}$

Lambertsche W-Funktion (inverse Fkt. zu $y = xe^x$)

"Product Log"