Karlsruhe Institute of Technology

SoSe 2021

Classical Theoretical Physics II

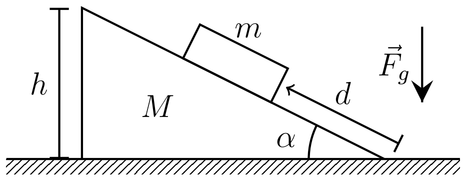
Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

Exercise Sheet 2

Issue: 23.04. – Submission: 30.04. @ 10:00 Uhr – Discussion: 03./04.05.

Exercise 1: Sliding wedge, take two **7 points**

Consider the same situation as in exercise 1 from the first exercise sheet: a block is sliding frictionlessly on a wedge with inclination α, which itself slides without friction on a horizontal surface. Here, we want to use the Lagrange formalism to see how it simplifies the solution.



(a) Construct the Lagrangian in terms of Cartesian coordinates (x_M, y_M) and (x_m, y_m) for the positions of the wedge and the block, respectively.

(b) The Cartesian coordinates are not independent of each other. Find the geometric constraints that the coordinates are subject to. You can express the constraints directly in terms of coordinates (instead of their derivatives). Use them to express the Lagrangian in terms of only two independent generalised coordinates. *Hint:* The distance d between the block and the lower right edge of the wedge is a useful choice here.

(c) Derive the Euler-Lagrange equations for the block and the wedge.

(d) Solve the Euler-Lagrange equations for the initial conditions that both the block and the wedge are at rest at $t = t_0$.

(e) Which results do you expect for $\ddot{x}_M$ and $\ddot{d}$ in the limits $\alpha \to 0$ and $\alpha \to \pi/2$? Does the result in (d) indeed tend to those limits?

(f) What is $\ddot{d}$ in the limit $m \ll M$ (for $\alpha \approx 30°$)?

(g) Calculate the time that its take for the block to travel from the upper edge of the wedge at height h to the surface. Compare your result to the result from exercise 1 from the first exercise sheet.

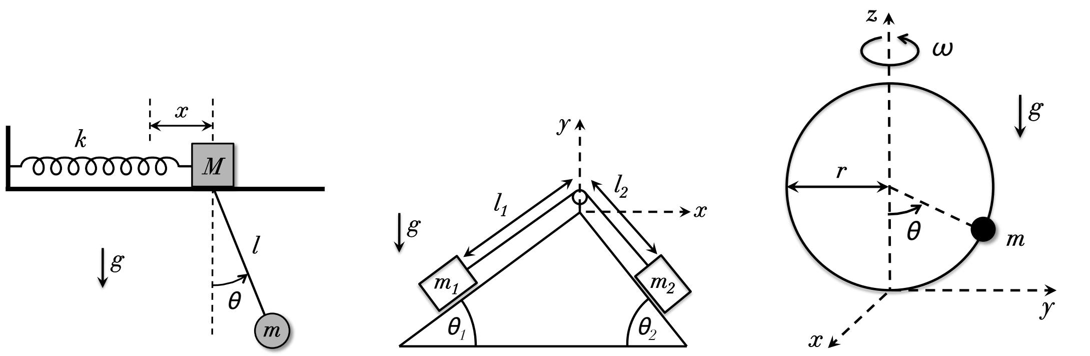
Exercise 2: Lagrangians for various systems **6 points**

Find the Lagrangian L for the following systems. Write $L(\vec{q}_i(t), \dot{\vec{q}}_i(t), t)$ first in terms of Cartesian coordinates. Then find the constraints on the systems and write the Lagrangian in terms of independent coordinates.

(a) Fig. 1(a): A mass M moves frictionlessly on a horizontal surface and is attached to a spring with spring constant k and displacement x. There is a mathematical pendulum of mass m and length l attached to the first mass M. The motion of the whole system happens in a two-dimensional plane.

(b) Fig. 1(b): Two masses m_1 and m_2 move under the influence of gravity and without any friction on a wedge. They are connected via a massless rope of length $l = l_1 + l_2$. The angles θ_1 and θ_2 of the wedge are fixed.

(c) Fig. 1(c): A bead with mass m glides frictionlessly along a circular wire with radius r. The wire is flat and rotates around the z axis with an angular frequency of ω.

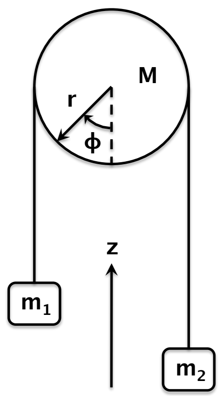


(a) (b) (c)

Figure 1: Three different systems.

Exercise 3: Atwood's machine **7 points**

Atwood's machine consists of two weights with masses m_1 and m_2, connected via a massless rope of length ℓ around a pulley with radius r and mass M. As one of the masses moves downward, the other mass is pulled upwards. The pulley rotates accordingly, so that the rope does not slip. The aim of this problem is to practice with the method of Lagrange multipliers.



(a) The kinetic energy of the rotating pulley is given by

$$T_{\text{pulley}} = \tfrac{1}{4} M r^2 \omega^2 \,, \tag{3.1}$$

where $\omega = \dot{\phi}$ is the angular velocity. (This will be derived later in the course.) Show that the no-slipping condition implies that $\omega = \dot{z}_1 / r$, where z_1 is the vertical position of mass m_1.

(b) Construct the Lagrangian $L(z_1, \dot{z}_1, z_2, \dot{z}_2)$, where z_2 is the vertical position of mass m_2. (The centre of the pulley remains at a fixed position.)

(c) Give the constraint that the rope has a fixed length ℓ in the form $f(z_1, z_2) = 0$.

(d) Derive the Euler-Lagrange equations from $L_{\text{tot}} = L(z_1, \dot{z}_1, z_2, \dot{z}_2) + \lambda f(z_1, z_2)$.

(e) Eliminate the Lagrange multiplier λ from the Euler-Lagrange equations and impose the constraint $f(z_1, z_2) = 0$ to derive the acceleration of mass m_1,

$$\ddot{z}_1 = \frac{(m_2 - m_1)g}{m_1 + m_2 + M/2} \,. \tag{3.2}$$

(f) The total force acting on mass m_2 is the vector sum of the tension in the rope (upwards) and the gravitational force (downwards). Compute the tension in the rope attached to mass m_2.

(g) Can you give the physical interpretation of the Lagrange multiplier λ?