Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

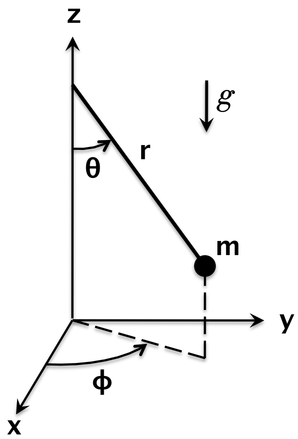
Exercise Sheet 3

Issue: 30.04. – Submission: 07.05. @ 10:00 Uhr – Discussion: 10./11.05.

Exercise 1: Spherical Pendulum

5 points

A spherical pendulum consists of a massless rod of length r, which is attached to a fixed point at one end and has a mass m on the other end. The pendulum is free to move, under the influence of gravity, in two directions around the fixed point. The position of the mass is thus described by spherical coordinates (θ, ϕ).



(a) Construct the Lagrangian for this system.

(b) The Lagrangian is independent of time t. Find the simplest choice of X and Ψ_i such that the Lagrangian is invariant under the corresponding infinitesimal transformation

$$t' = t + \epsilon\, X(\{q_j\}, t)\,, \qquad q_i' = q_i + \epsilon\, \Psi_i(\{q_j\}, t)\,, \tag{1.1}$$

where $\epsilon \ll 1$. What is the corresponding conserved quantity? Calculate its expression, starting from the general formula for the conserved Noether charge,

$$I = \sum_i \frac{\partial L}{\partial \dot{q}_i} (\Psi_i - X \dot{q}_i) + L\, X\,, \tag{1.2}$$

and describe its physical interpretation.

(c) The Lagrangian is also independent of the coordinate ϕ. What is the related symmetry transformation? Calculate the conserved quantity from Eq. (1.2). What is the physical interpretation of this conserved quantity?

(d) Derive the Euler-Lagrange equations. Which equation leads to the conserved quantity of the preceding question?

(e) Suppose that $\theta = \theta_0$ is constant. Show that the pendulum revolves around the vertical axis with constant angular velocity

$$\dot{\phi}_0 = \sqrt{\frac{g}{r \cos\theta_0}}\,. \tag{1.3}$$

Exercise 2: Conserved quantities in various potentials $\boxed{\textbf{6 points}}$

Consider a particle of mass m with position $\vec{r} = (r_1, r_2, r_3)$, whose motion is described by the Lagrangian $L = \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r}, t)$. We will consider various types of potentials $U(\vec{r}, t)$, in order to practice finding transformations that leave the action invariant and to compute the associated conserved quantities.

(a) Suppose that the potential has the form $U(\vec{r}, t) = U(\vec{r} - \vec{v}_0 t)$, where $\vec{v}_0$ is a constant vector. What infinitesimal transformation, of the form

$$t' = t + \epsilon\, X(\{r_j\}, t)\,, \qquad r'_i = r_i + \epsilon\, \Psi_i(\{r_j\}, t)\,, \tag{2.1}$$

leaves the Lagrangian (and thus the action) invariant? Show that the associated conserved quantity is

$$E(t) - \vec{p}\cdot\vec{v}_0 = \text{const}\,, \tag{2.2}$$

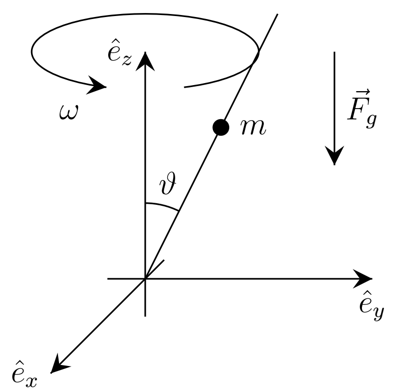
where the energy $E(t) = T + U = \frac{1}{2}m\dot{r}_i^2 + U$ is actually *not* constant.

(b) Suppose instead that the potential has the form $U(\vec{r}, t) = -F_0 r_3$, where F_0 is a constant force. Obviously, the Lagrangian does not depend explicitly on time, so $t \to t + \epsilon$ is a symmetry of the action and energy is conserved. Find additional symmetry transformations of the action, of the form of Eq. (2.1). Show that they lead to momentum conservation in the r_1, r_2 plane and to conservation of angular momentum around the r_3 axis. Can you generalise this problem to the potential $U(\vec{r}, t) = -\vec{F}_0 \cdot \vec{r}$?

(c) Let $U(\vec{r}, t) = e\,\phi(\vec{r}, t) - \frac{e}{c}\vec{A}(\vec{r}, t)\cdot\dot{\vec{r}}$ for a particle with charge e. Choose the potentials as $\phi = 0$ and $\vec{A} = B_0\, r_1\, \hat{e}_2$, which corresponds to a constant magnetic field $\vec{B} = B_0\, \hat{e}_3$. Find all symmetry transformations of the Lagrangian and calculate the corresponding conserved quantities as functions of r_i and $\dot{r}_i$.

Exercise 3: Bead on an angled wire $\boxed{\textbf{5 points}}$

A bead of mass m, idealised as a point mass, moves frictionlessly on a wire, which rotates with angular velocity ω and constant inclination α around the z-axis (see sketch). The system is subject to Earth's homogenous gravitational force $\vec{F}_g = -mg\hat{e}_z$.



(a) Construct the Lagrangian for this system using suitable generalised coordinates.

(b) Compute the conserved quantity that is generated by the fact that the Lagrangian does not depend explicitly on time.

(c) Is the conserved quantity in the previous question equal to the energy of the bead m? Provide an explanation in either case.

(d) Find and solve the equations of motion for the initial conditions $r(t=0) = r_0 > 0$ and $\dot{r}(t=0) = 0$. Here, $r(t)$ denotes the distance of the bead from the origin.

(e) How does the system behave for very small or very large values of α (i.e., for $\alpha \approx 0$ and $\alpha \approx \frac{\pi}{2}$)? How does the angular velocity ω influence the behaviour? For which value of ω does the system change from one behaviour to the other?

Exercise 4: Snell's law of refraction $\boxed{\textbf{6 points}}$

Fermat's principle says that light propagates along the path along which it takes the shortest time between two points. The goal of this exercise is to derive Snell's law of refraction from this principle.

(a) Show that the time which it takes for light to propagate in a plane from a point P with coordinates (x_1, y_1) to a point Q with coordinates (x_2, y_2), while in a medium[1] with an index of refraction $n(x, y)$, is given by

$$T[y(x)] = \int_P^Q \mathrm{d}t = \int_{x_1}^{x_2} \mathrm{d}x\, F(y(x), y'(x), x)\,, \tag{4.1}$$

where

$$F(y(x), y'(x), x) = \frac{n(x,y)}{c}\sqrt{1 + y'(x)^2}\,. \tag{4.2}$$

(b) Draw an analogy between Fermat's principle and the principle of minimal action to find the Euler-Lagrange equation for the trajectory $y(x)$ that minimises the time functional $T[y(x)]$. Show that this implies

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'(x)\, n(x,y)}{\sqrt{1 + y'(x)^2}}\right) = \frac{\partial n(x,y)}{\partial y}\sqrt{1 + y'(x)^2} \tag{4.3}$$

with F from Eq. (4.2).

[1] The speed of light in a medium is given by $v = c/n$.

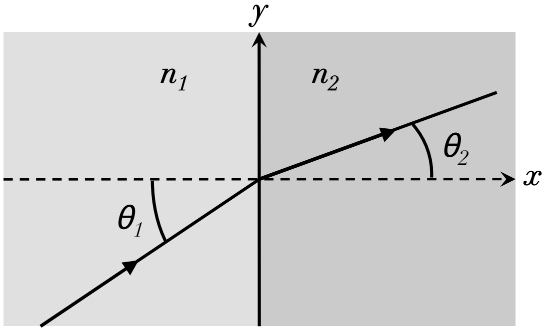


Figure 1: Refraction of light at a boundary surface between two regions with different, constant indices of refraction

(c) Show that in a medium with constant index of refraction $n(x, y) = n_0$ light propagates along a straight line.

(d) Consider the situation in Fig. 1, where

$$n(x, y) = n(x) = \begin{cases} n_1 & \text{für } x < 0 \\ n_2 & \text{für } x > 0 \end{cases} \tag{4.4}$$

with n_1 and n_2 being different constants. Use the solution from question (c) in both regions and use Eq. (4.3) to derive Snell's law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \,. \tag{4.5}$$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WANN? Mittwoch, 05.05. 18:00 Uhr
WO? Zoom, siehe: fachschaft.physik.kit.edu
eine Veranstaltung des Mentorenprogramms


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