Classical Theoretical Physics II

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Exercise Sheet 4

Issue: 07.05. – Submission: 14.05. @ 10:00 Uhr – Discussion: 17./18.05.

Exercise 1: Particle on a paraboloid **9 points**

We have seen in the lecture that we can deal with constraints in different ways if they are given in the form $f_i(q_1, \ldots, q_N, t) = 0$, $i \in \{1, \ldots, k\}$. One option is to first solve the constraints for the generalised coordinates and to use them to eliminate the dependent coordinates in the Lagrangian. Then we can derive the Euler-Lagrange equations for the independent coordinates. This is called the *Lagrange formalism of the second kind.* Another option is to introduce Lagrange multipliers, to construct an extended Lagrangian $L_{\text{tot}} = L + \sum_{i=1}^{k} \lambda_i f_i$ and to derive the Euler-Lagrange equations for L_{tot}. This is called the *Lagrange formalism of the first kind.* As we have seen in the lecture, with this we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\text{tot}}}{\partial \dot{q}_i} = \frac{\partial L_{\text{tot}}}{\partial q_i} \qquad \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_i}\,, \tag{1.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_{\text{tot}}}{\partial \dot{\lambda}_i} = \frac{\partial L_{\text{tot}}}{\partial \lambda_i} \qquad \Leftrightarrow 0 = f_i\,. \tag{1.2}$$

The additional terms $Z_{i,j} = \lambda_j \frac{\partial f_j}{\partial q_i}$ on the right-hand side of the first equation are the components of the constraining forces that result from the constraints.

As an example, we now want to consider a particle of mass m. It slides without friction under the influence of a homogeneous gravitational field ($\vec{F}_g = -mg\hat{e}_z$) on the inside of a paraboloid of revolution, which is described by the equation $z = (x^2 + y^2)/a$. It makes sense to use cylindrical coordinates (ρ, φ, z) with $\rho = \sqrt{x^2 + y^2}$ and $\tan\varphi = y/x$ to parametrise the position of the particle.

(a) Draw a sketch of the system.

(b) Use the Lagrange formalism of the first kind to derive the equations of motion. Combine the different equations to derive an equation for $\rho(t)$ which only depends on this variable and its derivatives as dynamical variables. Give the equations which express $\varphi(t)$ and $z(t)$ in terms of $\rho(t)$. What are the constraining forces that act on the particle? *Hint:* One of the Euler-Lagrange equations yields a conserved quantity. Use it to eliminate one of the variables. (You do not need Noether's theorem for that in this case.)

(c) Derive the equations of motion for the same choice of coordinates using the Lagrange formalism of the second kind.

(d) Show that the particle moves on a circular trajectory on the inside of the paraboloid in the plane given by $z = h$ if the particle starts with an angular velocity of $|\omega| = \sqrt{2g/a}$ in horizontal direction. *Hint:* You only have to check that this trajectory is a solution of the equations of motion.

(e) Calculate the constraining forces that act on the particle on the trajectory from (d) using your results from (b).

Exercise 2: Generalising Noether's theorem — 4 points

In the lectures we have derived a general expression for conserved quantities from the invariance of the action under infinitesimal transformations of the form

$$q_i' = q_i + \epsilon \Psi_i(q,t)\,, \qquad t' = t + \epsilon X(q,t)\,, \tag{2.1}$$

where we use the shorthand $q = \{q_j\}$. Here we want to consider the case where the Lagrangian does change, but only by a total time derivative:

$$L(q, \mathrm{d}q/\mathrm{d}t, t) = L(q', \mathrm{d}q'/\mathrm{d}t', t') + \epsilon \frac{\mathrm{d}f(q',t')}{\mathrm{d}t'}\,, \tag{2.2}$$

where f is a function of the positions and the time.

(a) Argue that the additional term $\epsilon \mathrm{d}f(q',t')/\mathrm{d}t'$ does not change the Euler-Lagrange equations.

(b) Show that the invariant quantity in this case is given by

$$I = \sum_i \frac{\partial L}{\partial \dot{q}_i}(\Psi_i - X\dot{q}_i) + L\,X + f(q,t)\,. \tag{2.3}$$

Hint: Follow the same steps as in the lecture.

Exercise 3: Galilean transformations — 5 points

Let us look at an (infinitesimal) Galilei transformation $q_i \to q_i + \epsilon w_i t$.

(a) What is the physical interpretation of this transformation and of the w_i?

(b) Derive an expression for the conserved quantity I for a free particle of mass m, that follows from the invariance of the action under Galilei transformations. *Hint:* Identify Ψ_i, X, and f in Eq. (2.3).

(c) Show that this implies the conservation of

$$\chi_i = m q_i - p_i t\,, \tag{3.1}$$

for $i \in \{1,2,3\}$. ($p_i = m\dot{q}_i$ denotes the momentum).

(d) For a free particle the energy (one equation), momentum (three equations), and angular momentum (three equations) are conserved. Additionally, the three quantities of Eq. (3.1) are conserved. Thus, we have ten integrals of motion in total that characterise the physics. But the movement of a free particle is completely specified by six boundary conditions (e.g., the position and velocity at $t = 0$). Why does the existence of ten integrals of motion not over-determine the system?

(e) Consider a more general Lagrangian

$$L = \frac{m}{2} f(\dot{\vec{q}}^2)\,, \tag{3.2}$$

where f is a general, differentiable function which depends only on the square of the norm of the velocity. Show that the form of f is completely determined by demanding that the action is invariant under (infinitesimal) Galilei transformations.