Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

Exercise Sheet 5

Issue: 14.05. – Submission: 21.05. @ 10:00 Uhr – Discussion: 31.05./01.06.

Exercise 1: Similarity transformations — 6 points

Consider a particle of mass m with position $\vec{r} = (r_1, r_2, r_3)$, whose motion is described by the Lagrangian $L = \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r})$.

(a) Suppose that $U(\lambda\vec{r}) = \lambda^{-2}U(\vec{r})$. Find a similarity transform of the coordinates r_i and time t (i.e., a rescaling by a factor λ) which leaves the action (but not the Lagrangian) invariant.

(b) In contrast to the situation discussed in the lecture, here the transformed action S' is not only proportional but actually identical to the original action S. Therefore, Noether's theorem yields an invariant. Write the similarity transform in infinitesimal form as

$$t' = t + \epsilon\, X(\{r_j\}, t)\,, \qquad r'_i = r_i + \epsilon\, \Psi_i(\{r_j\}, t)\,, \tag{1.1}$$

and show that the associated invariant is

$$\vec{p} \cdot \vec{r} - 2Et = \text{const}\,. \tag{1.2}$$

(c) Now consider the potential $U(\vec{r}) = -k/r^2$, with $k > 0$. Confirm that this is a special case of the previous question, so that Eq. (1.2) holds true. Argue that the angular momentum $\vec{M} = \vec{r} \times \vec{p}$ is also conserved. Show that the third invariant, energy, can be written as

$$E = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{k}{r^2} \tag{1.3}$$

(d) Finding all these conserved quantities is very useful for determining the trajectory in a relatively easy way, without solving differential equations. Indeed, combine Eqs. (1.2) and (1.3) to derive the trajectory in the radial direction

$$r(t) = \sqrt{\frac{(2Et + \text{const})^2 + M^2 - 2mk}{2mE}}\,. \tag{1.4}$$

Hint: show that $\vec{p} \cdot \vec{r} = mr\dot{r}$ holds.

Exercise 2: Mechanical similarity

3 points

Consider a particle of mass m in a potential of homogeneous degree n,

$$U(\lambda x) = \lambda^n U(x)\,. \tag{2.1}$$

In the one-dimensional case this implies that $U(x) = U_0\, x^n$. Assume that the potential has a minimum, i.e. $U_0 > 0$ and $n = 2, 4, 6, \ldots$, and that the energy E of the particle is larger than the minimum U_0. Then the particle oscillates between two turning points $x_A(E)$ and $x_B(E)$, given by the condition $E = (x_{A,B})$.

(a) Show that the period of oscillation T is proportional to $E^{1/n-1/2}$. *Hint:* make a suitable change of variables, without actually carrying out the integral.

(b) Using the previous result, derive the statement of mechanical similarity,

$$\frac{T_1}{T_2} = \left(\frac{L_1}{L_2}\right)^{1-n/2}\,, \tag{2.2}$$

between two orbits with periods T_i and length scales $L_i \sim (x_{A,B})_i$.

Exercise 3: Motion in different potentials

3 Punkte

Describe the motion of a particle in the potentials shown in Fig. 1 qualitatively (finite/infinite, turning points). Focus in particular on the different possible types of motion depending on the energy of the particle. The first diagram already contains markers for the relevant energy levels. Draw a sketch for the other two potentials and mark the relevant energy levels.

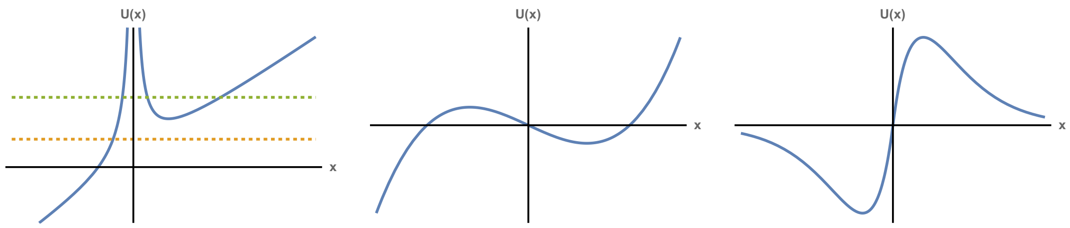

Figure 1: Different potentials

Exercise 4: Reaching the origin

6 points

For the one-dimensional motion of a particle of mass m, the time it takes to travel from some fixed initial coordinate x_0 to a time-dependent coordinate $x(t)$ can be written as the integral

$$t - t_0 = \pm\sqrt{\frac{m}{2}} \int_{x_0}^{x(t)} \frac{\mathrm{d}x}{\sqrt{E - U(x)}}\,, \tag{4.1}$$

where $x(t_0) = x_0$. In this exercise, let the energy E be zero and $U(x) \leq 0$. Also, let the initial coordinate be positive, and let the initial velocity be negative.

(a) What is the correct sign ($\pm$) in Eq. (4.1) for these boundary conditions?

Consider the potential

$$U(x) = -C\,|x|\,, \tag{4.2}$$

where $|x|$ is the absolute value of x and C is a positive constant.

(b) Sketch the potential $U(x)$. Indicate in the sketch the coordinates x_0 and $x(t)$ at some time $t > t_0$, such that $x(t)$ is still positive.

(c) Perform the integral in Eq. (4.1) with the given $U(x)$.

(d) Solve the result of the previous question for $x(t)$ in terms of C, m, x_0, t_0 and t.

(e) How long does it take the particle to reach the origin $x(t) = 0$?

(f) Repeat questions (b) – (e) for the potential $U(x) = -C\,x^2$ with $C > 0$.