Classical Theoretical Physics II

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Exercise Sheet 6

Issue: 21.05. – Submission: 04.06. @ 10:00 Uhr – Discussion: 07./08.06.

Exercise 1: Three-dimensional harmonic oscillator — 8 points

Consider the three-dimensional generalisation of the harmonic oscillator where two masses m_1 and m_2 interact via the potential $U(\vec{r}_1, \vec{r}_2) = \alpha(\vec{r}_1 - \vec{r}_2)^2$.

(a) Show that the motion of the two particles can be divided into the motion of the centre of mass (the motion of a free particle with mass m) and the motion of a particle with mass μ in the central potential $U(\vec{r}) = U(r)$. Determine m and μ.

(b) The motion of the centre of mass is not very interesting. Argue that the motion of the particle in the potential $U(r)$ occurs in a plane, so that the trajectory of the particle can be described in polar coordinates r and φ. Show also that the energy can be written as

$$E = \frac{\mu}{2}\dot{r}^2 + U_{\text{eff}}(r)\,. \tag{1.1}$$

Find and sketch $U_{\text{eff}}(r)$.

(c) Find the minimum $U_{\text{eff,min}}$ of the effective potential and determine the turning points r_A and r_B under the assumption that the energy of the particle fulfils $E > U_{\text{eff,min}}$.

(d) We can find the expression

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\dot{\varphi} = \frac{\mathrm{d}r}{\mathrm{d}\varphi}\frac{M}{\mu r^2} \tag{1.2}$$

for the velocity of the particle, where M is the norm of the angular momentum. Show that for a certain choice of integration constants we can write

$$\varphi = \frac{1}{2}\arccos\left(\frac{\frac{M^2}{\mu r^2} - E}{\sqrt{E^2 - \frac{2\alpha M^2}{\mu}}}\right)\,. \tag{1.3}$$

Hint: Use the integral

$$\int \frac{\mathrm{d}z}{\sqrt{az^2 + bz + c}} = \frac{-1}{\sqrt{-a}}\arccos\left(-\frac{b + 2az}{\sqrt{b^2 - 4ac}}\right) \tag{1.4}$$

and the substitution $z = 1/r^2$.

(e) Calculate the change of the angle $\Delta\varphi$, that arises when the particle oscillates from $r = r_A$ to $r = r_B$. Use this to determine whether the trajectory is open or closed.

(f) Invert the result from Eq. (1.3) to find the trajectory $r(\varphi)$. Which value of φ corresponds to the minimal radius $r = r_A$?

Exercise 2: Overcoming angular momentum **9 points**

Consider a particle of mass m which moves subject to the three-dimensional potential

$$U(\vec{r}) = -\alpha r^{-n} \,. \tag{2.1}$$

The angular momentum of the particle should be non-vanishing, $M \neq 0$.

(a) What are the conditions on α and n in order for the particle to be able to reach the origin? Distinguish the different possible cases and sketch the effective potential in these cases. How large does the total energy have to be such that a particle coming from infinity can reach the origin?

(b) Is the number of orbits before reaching the origin finite or infinite in those cases from (a) where the origin can be reached?

(c) Assume that the conditions from (a) are fulfilled. Show that the time required to reach the origin is always finite, while the velocity and the angular velocity go to infinity.

Hint: The integrals that appear in (b) and (c) do not have to be calculated exactly, but it is sufficient to discuss their behaviour for small r (distance to the origin).