Karlsruhe Institute of Technology

SoSe 2021

Classical Theoretical Physics II

Lecture: Prof. Dr. K. Melnikov – Exercises: Dr. A. Behring

Exercise Sheet 7

Issue: 04.06. – Submission: 11.06. @ 10:00 Uhr – Discussion: 14./15.06.

Exercise 1: Precession of the perihelion **10 points**

Einsteins' general theory of relativity predicts that the orbits of the planets are not stable ellipses as predicted by Newton, but rather ellipses for which the perihelion (i.e. the point on the orbit closest to the sun) slowly rotates around the sun. The correct prediction of the size of this effect for the orbit of the planet Mercury was the first triumph of general relativity, and helped to establish that theory as the successor to Newtonian gravitation.

The leading effect of the relativistic corrections, can be modelled as an additional term in the gravitational potential which decreases as r^{-2}, such that the potential becomes

$$U(\vec{r}) = -\frac{k}{r} + \frac{C}{r^2}\,, \tag{1.1}$$

where k is the same as in Newtonian gravity, and C parametrises the relativistic correction.

(a) Write down the Lagrangian for an object with mass m moving in this three-dimensional potential. What are the conserved quantities? Explain why this implies that the motion takes place on a two-dimensional plane that is naturally described by polar coordinates r and φ.

(b) For an object moving in a central potential, φ as a function of r can be written as

$$\varphi = \int \frac{\mathrm{d}r}{r^2\sqrt{\frac{2mE}{M^2} - \frac{2mU(r)}{M^2} - \frac{1}{r^2}}}\,, \tag{1.2}$$

where M is the angular momentum of the object and E is its energy. Show that this implies that the shape of the orbit is given by

$$r = \frac{\rho(1-\epsilon^2)}{1+\epsilon\cos(\alpha\varphi)}\,, \tag{1.3}$$

where ρ, ϵ, and α are functions of E, M, m, k, and C. The expression for α is

$$\alpha = \sqrt{1 + \frac{2mC}{M^2}}\,. \tag{1.4}$$

What are the expressions for ρ and ϵ?

Hint: Follow the same steps as for the derivation of the Kepler orbits presented in the lecture. Use the integral

$$\int \frac{\mathrm{d}z}{\sqrt{az^2 + bz + c}} = \frac{-1}{\sqrt{-a}} \arccos\left(-\frac{b + 2az}{\sqrt{b^2 - 4ac}} \right), \tag{1.5}$$

and the substitution $z = 1/r$.

(c) Make a sketch of the orbit for values of α close to one. Show α, ρ, and ϵ in the sketch. What is the essential difference between the cases of $\alpha = 1$ and $\alpha \neq 1$?

(d) Assume in the following that C is much smaller than all other scales in the problem. Derive that the rate of the precession of the perihelion, i.e., the change in the angular position of the perihelion for each orbit of the planet, is

$$\delta\varphi \approx -2\pi \frac{mC}{M^2}. \tag{1.6}$$

(e) Einsteins' theory predicts that the constant C is given by

$$C = -3\, \frac{G^2\, m\, m_\odot^2}{c^2} \tag{1.7}$$

where G is Newtons' gravitational constant, $m_\odot$ is the mass of the sun, and c the speed of light. Show that the rate of the precession of the perihelion may be expressed as

$$\delta\varphi \approx \frac{6\pi G m_\odot}{c^2(1-\epsilon^2)\rho} \tag{1.8}$$

Hint: You may use the Newtonian expressions for k, ρ, and ϵ:

$$k = G\, m\, m_\odot\,, \qquad \rho = -\frac{G\, m\, m_\odot}{2E}\,, \qquad \epsilon \approx \sqrt{1 + \frac{2EM^2}{mk^2}}\,. \tag{1.9}$$

(f) The measured value of the rate of perihelion precession of Mercury is 5600 arcseconds per century (an arcsecond or is 1/3600 of a degree), but most of this effect is due gravitational effects from the other planets, well described by Newtonian physics. How big is the relativistic contribution to the precession of the perihelion of Mercury?
Hint: The planet Mercury has $\epsilon = 0.206$, $\rho = 57.9 \times 10^9\,\mathrm{m}$, orbital period $\tau = 88.0$ (earth-)days, and $m = 3.30 \times 10^{23}\,\mathrm{kg}$. Additionally, $m_\odot = 1.99 \times 10^{30}\,\mathrm{kg}$, $G = 6.67 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ and $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$.

Exercise 2: The Runge-Lenz vector **10 points**

We have shown in the lecture that the Runge-Lenz vector $\vec{A}$ is conserved in the Kepler problem. The Runge-Lenz vector is defined as

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}\,, \tag{2.1}$$

where $\vec{p}$ is the momentum, $\vec{L}$ is the angular momentum, $k = GMm$ and $\hat{r} = \vec{r}/r$ is the unit vector in radial direction.

(a) Sketch an elliptical orbit and indicate the direction of the Runge-Lenz vector at different points along the orbit.

(b) We introduce the angle θ between $\vec{A}$ and $\vec{r}$ such that $\vec{A} \cdot \vec{r} = |\vec{A}||\vec{r}|\cos\theta$. Equate this expression to the explicit expression for $\vec{A} \cdot \vec{r}$ to derive the formula

$$r = \frac{L^2}{mk + A\cos\theta}\,, \tag{2.2}$$

where $L = |\vec{L}|$.

(c) Compare Eq. (2.2) with the formula for an elliptic orbit

$$r = \frac{(1-\epsilon^2)\rho}{1+\epsilon\cos\theta} \tag{2.3}$$

to express A in terms of the excentricity ϵ.

(d) If there are deviations from a $1/r$ potential, the Runge-Lenz vector is no longer conserved. Show that the time derivative of the Runge-Lenz vector for a general central potential $U(r) = -\frac{k}{r} + \delta U(r)$ is given by

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}t} = f(r)\,\hat{r} \times \vec{L}\,, \tag{2.4}$$

where $f(r) = -\frac{\mathrm{d}\delta U(r)}{\mathrm{d}r}$.
Hint: Use the vector identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ (BAC-CAB rule).

(e) Use the conservation of angular momentum, to express the time derivative as a derivative with respect to the angle and to obtain the formula

$$\frac{\mathrm{d}\vec{A}}{\mathrm{d}\theta} = -f(r)mr^2\hat{L} \times \hat{r}\,, \tag{2.5}$$

where $\hat{L} = \vec{L}/L$ is the unit vector in $\vec{L}$ direction.

(f) Now we consider the $1/r^2$ perturbation from the previous problem so that $f(r) = \frac{2C}{r^3}$ holds. Show that the change of the Runge-Lenz vector after one complete orbit is given approximately by

$$\Delta\vec{A} = \frac{-2\pi Cm}{L^2}\hat{L} \times \vec{A}\,. \tag{2.6}$$

Hint 1: Choose the coordinate system such that you can describe $\vec{r}$ in polar coordinates ($\vec{r} = r(\cos\theta\hat{e}_x + \sin\theta\hat{e}_y)$) and that $\hat{L} = L\hat{e}_z$.
Hint 2: $\int_0^{2\pi} \cos\theta\sin\theta = 0$ and $\int_0^{2\pi} \cos^2\theta = \pi$.

(g) Compare Eq. (2.6) with the formula for the speed of the perihelion precession from the previous exercise,

$$\Delta\theta = \frac{-2\pi m C}{L^2}\,. \tag{2.7}$$

How do you interpret this agreement? How does this fit to the sketch from subproblem (a)?